Algebraic Models for Homotopy Types III
Algebraic Models in p-Adic Homotopy Theory

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Review

Polynomial De Rham functor Ω^*_{pl} from simplicial sets to commutative differential graded algebras

- Defined by $\Omega^*_{\mathrm{pl}} X_\bullet = \mathrm{Hom}_{\Delta^{\mathrm{op}}}(X_\bullet, \nabla^*_\bullet)$
- Has an adjoint $U_\bullet A = \mathcal{CA}(A, \nabla^*_\bullet)$
- Quillen adjunction

$\mathbb{P}\langle x\rangle \to \Omega^*_{\mathrm{pl}}(K(\mathbb{Q}, n))$ is a weak equivalence.

$$K(\mathbb{Q}, n) \to U_\bullet(\mathbb{P}\langle x\rangle) = n\text{-cycles in} \nabla^*_\bullet$$

is a weak equivalence.

Ω^*_{pl} and $U_\bullet$ are inverse equivalences on the homotopy categories of finite type simply connected spaces and finite type cohomologically simply connected CDGAs.

Outline

Theorem

The p-adic homotopy theory of finite type simply connected spaces is equivalent to the homotopy theory of finite type cohomologically simply connected spacelike E_∞ $\bar{\mathbb{F}}_p$-algebras.

Mandell, Michael A. E_∞ algebras and p-adic homotopy theory, 2001.

1. p-Adic homotopy theory
2. Cochains and forms
3. E_∞ algebras
4. The Quillen adjunction and the theorem

p-Adic Homotopy Theory

Definition

A p-adic equivalence is a map $X \to Y$ that induces an isomorphism on $\mathbb{F}_p$ homology $H_*(-;\mathbb{F}_p)$, or equivalently, $\mathbb{F}_p$ cohomology $H^*(-;\mathbb{F}_p)$.

Theorem (mod p Whitehead Theorem)

If X and Y are finite type simply connected spaces, then a map $X \to Y$ induces an isomorphism on $H_*(-;\mathbb{F}_p)$ if and only if it induces an isomorphism on $H_*(-;\mathbb{Z}_p^\wedge)$ if and only if it induces an isomorphism on $\pi_* \otimes \mathbb{Z}_p^\wedge$.

Theorem (Bousfield $H_*(-;\mathbb{F}_p)$-Local Model Structure)

The category of simplicial sets admits a model structure where

- *The weak equivalences are the p-adic equivalences*
- *The cofibrations are the injections*
- *The fibrations are the maps with the RLP with respect to the acyclic cofibrations.*

p-Complete Spaces and p-Completion

A finite type simply connected space is p-complete if and only if it is weakly equivalent to a fibrant object in the Bousfield $H_*(-;\mathbb{F}_p)$-local model structure. (This is actually a theorem rather than a definition)

The p-completion map $X \to X^\wedge_p$ is the initial map in the homotopy category from X to a p-complete space.

It is also the unique (up to unique isomorphism) map in the homotopy category that is a p-adic equivalence from X to a p-complete space.

Theorem (Characterization of p-complete spaces)

If X is simply connected and finite type, then
X is p-complete
$\iff$ *π_*X is p-complete ($\pi_*X \cong \lim \pi_*X/p^n$)*
$\iff$ *$H_*(X;\mathbb{Z})$ is p-complete*

Polynomial Differential Forms

Try to construct p-adic differential forms

Free commutative differential graded $\mathbb{Z}^\wedge_p$-algebra

$$\mathbb{P}X = \mathbb{Z}^\wedge_p \oplus X \oplus (X \otimes_{\mathbb{Z}^\wedge_p} X)/\Sigma_2 \oplus \cdots$$

Polynomial forms on an n-simplex

$$\nabla^*_n = \mathbb{P}\langle t_0, \ldots, t_n, dt_0, \ldots, dt_n \mid \sum t_i = 1, \sum dt_i = 0\rangle$$

Problem

∇^*_1 is not acyclic. $t^{p-1}dt$ is a cycle but not a boundary: $d(t^p) = pt^{p-1}dt$.

You can try to fix this using "divided powers" (include $\frac{1}{p}t^p$)
[Grothendieck, "Pursuing Stacks", §94]

But nothing like this will work

Contradiction

Suppose we could form some version Ω^*_{pl} of the polynomial DeRham complex over $\mathbb{Z}$ or $\mathbb{Z}^\wedge_p$.

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow \\ \Delta^1 & \longrightarrow & S^1 \end{array} \qquad \begin{array}{ccc} \Omega^*_{\mathrm{pl}} S^1 & \longrightarrow & \Omega^*_{\mathrm{pl}} \Delta^1 \\ \downarrow & & \downarrow \\ \Omega^*_{\mathrm{pl}} \Delta^0 & \longrightarrow & \Omega^*_{\mathrm{pl}}(\{0,1\}) \end{array}$$

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0 S^1 & \longrightarrow & H^0\Delta^0 \oplus H^0\Delta^1 & \longrightarrow & H^0(\{0,1\}) & \longrightarrow & H^1 S^1 & \longrightarrow 0 \\ & \| & & \| & & \| & & \| & \\ 0 \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \end{array}$$

Steenrod Operations
$(p = 2)$

Steenrod squares $Sq^i \colon H^n(X;\mathbb{F}_2) \to H^{n+i}(X;\mathbb{F}_2)$

Characterized by following properties:

- Natural homomorphisms
- Commute with suspension isomorphism / boundary map
- If $|x| = i$ then $Sq^i x = x^2$ and $Sq^j x = 0$ for $j > i$ (unstable condition)

Calculation of $H^*(K(\mathbb{F}_2, n);\mathbb{F}_2)$. Further properties:

- $Sq^0 = \mathrm{id}$ and $Sq^i = 0$ for $i < 0$
- Cartan formula: $Sq^n(xy) = \sum (Sq^i x)(Sq^{n-i} y)$
- Adem relations: if $s < 2t$, $Sq^s Sq^t = \sum_i (s - 2i, t - a + i - 1) Sq^{s+t-i} Sq^i$

Construction of the Steenrod Operations

Multiplication on C^*X is not commutative but it is homotopy commutative and commutative up to “all higher homotopies”

$$E\Sigma_2 \otimes_{\Sigma_2} (C^*X)^{\otimes 2} \to C^*X$$

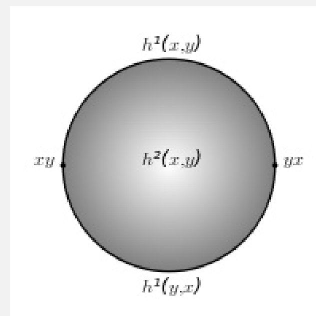
Working over $\mathbb{F}_2$,

$$dh^n(x,y) \equiv h^{n-1}(x,y) + h^{n-1}(y,x) + h^n(dx,y) + h^n(x,dy)$$

So for $dx \equiv 0 \mod 2$,

$$dh^n(x,x) \equiv h^{n-1}(x,x) + h^{n-1}(x,x) + 0 + 0 \equiv 0 \mod 2$$

$h^n(x,x)$ is a mod 2 cycle, represents $Sq^{|x|-n}x$.



Steenrod operations direct result of higher homotopy commutativity,
non-zero $\Longrightarrow$ **not** commutative w/o homotopy.

Sq^0 is very high level: Uses h^n

E_∞ Algebras

$$E\Sigma_n \otimes_{\Sigma_n} A^{\otimes n} \to A$$

Like a commutative differential graded algebra but instead of having just one possible way of multiplying n elements, it has a contractible complex $E\Sigma_n$ of n-ary products.

E_∞ algebras have Steenrod operations that satisfy

- Unstable condition
  $Sq^n x = x^2$ for $n = |x|$ and $Sq^n x = 0$ for $n > |x|$
- Cartan and Adem relations

But:

- No reason for Sq^0 to be the identity $(Sq^0 x = h^n(x,x))$
- No reason for Sq^i to be 0 for $i < 0$ $(Sq^i x = h^{n-i}(x,x))$

Steenrod algebra $\mathfrak{A}$ is a quotient of the algebra of all operations $\mathfrak{B}$
$\mathfrak{B}$ is the cohomologically graded full Araki-Kudo Dyer-Lashof algebra

Free E_∞ Algebras and Eilenberg–Mac Lane Spaces

In rational homotopy theory, we had $\Omega^*_{\mathrm{pl}}(K(\mathbb{Q}, n)) \simeq \mathbb{P}(\mathbb{Q}[n])$.

Free E_∞ $\mathbb{F}_p$-algebra

$$\mathbb{E}X = \mathbb{F}_p \oplus X \oplus E\Sigma_2 \otimes_{\Sigma_2} (X \otimes X) \oplus \cdots$$

$H^*(\mathbb{E}(\mathbb{F}_p[n]))$ is the free graded commutative $\mathfrak{B}$-algebra on a generator in degree n satisfying the unstable condition

$H^*(K(\mathbb{F}_p, n); \mathbb{F}_p)$ is the free graded commutative $\mathfrak{A}$-algebra on a generator in degree n satisfying the unstable condition

Ker $(\mathfrak{B} \to \mathfrak{A})$???

Free E_∞ Algebras and Eilenberg–Mac Lane Spaces II

The pushout $B_n = \mathbb{E}\langle x, y \mid dy = (1 - Sq^0)x\rangle$ where $|x| = n$.

Theorem

$\mathfrak{A} = \mathfrak{B}/(1 - P^0)$. The pushout

$$\begin{array}{ccc} \mathbb{E}(\mathbb{F}_p[n]) & \xrightarrow{1-P^0} & \mathbb{E}(\mathbb{F}_p[n]) \\ \downarrow & & \downarrow \\ \mathbb{F}_p\mathbb{E}(C\mathbb{F}_p[n]) & \longrightarrow & B_n \end{array}$$

is quasi-isomorphic to $C^*(K(\mathbb{F}_p, n); \mathbb{F}_p)$

The pushout $B_n = \mathbb{E}\langle x, y \mid dy = (1 - P^0)x\rangle$ where $|x| = n$.

For $p > 2$, Sq^0 is usually denoted P^0

The Homotopy Theory of E_∞ Algebras

The Model Structure on E_∞ Algebras

- Weak equivalences = quasi-isomorphisms
- Fibrations = surjections
- Cofibrations = formed by attaching cells of the form $\mathbb{E}\langle x\rangle \to \mathbb{E}\langle x, y \mid dy = x\rangle$

Spacial Realization

$C^*X_\bullet = \mathrm{Hom}_{\Delta^{\mathrm{op}}}(X_\bullet, C^*(\Delta^\bullet))$
Let $U_\bullet A = \mathcal{E}(A, C^*(\Delta^\bullet))$

$$\mathcal{S}(X_\bullet, U_\bullet A) \cong \mathcal{E}(A, C^*X_\bullet) = \mathcal{E}^{\mathrm{op}}(C^*X_\bullet, A)$$

C^* sends injections to surjections and preserves weak equivalences, so $C^*, U_\bullet$ is a Quillen adjunction.

The Unit Map for $K(\mathbb{F}_p, n)$

$$\mathrm{Ho}_p\,\mathcal{S}(X, Y) \xrightarrow{C_*} \mathrm{Ho}\,\mathcal{E}(C^*Y, C^*X) \cong \mathrm{Ho}_p\,\mathcal{S}(X, RU_\bullet C^*Y)$$

induced by unit map $Y \to RU_\bullet C^*Y = U_\bullet A$ for any cofibrant approximation $A \to C^*Y$.

For $B_n \xrightarrow{\simeq} C^*K(\mathbb{F}_p, n)$

$$\begin{array}{ccc} \mathbb{E}\bar{\mathbb{F}}_p[n] & \xrightarrow{1-P^0} & \mathbb{E}\bar{\mathbb{F}}_p[n] \\ \downarrow & & \downarrow \\ \mathbb{E}(C\bar{\mathbb{F}}_p[n]) & \longrightarrow & \bar{B}_n \end{array} \quad \overset{\bar{U}_\bullet}{\Longrightarrow} \quad \begin{array}{ccc} \bar{U}_\bullet\bar{B}_n & \longrightarrow & C^n(\Delta^\bullet; \bar{\mathbb{F}}_p) \simeq * \\ \downarrow & & \downarrow \\ K(\bar{\mathbb{F}}_p, n) & \xrightarrow[1??\phi]{} & K(\bar{\mathbb{F}}_p, n) \end{array}$$

So $U_\bullet B_n \simeq K(\mathbb{F}_p, n) \times K(\mathbb{F}_p, n-1)$.

To fix this, use E_∞ $\bar{\mathbb{F}}_p$-algebras: $\bar{U}_\bullet(\bar{B}_n) \simeq K(\mathbb{F}_p, n)$.

Algebraic Models for p-Adic Homotopy Theory

Theorem

If X and Y are finite type and simply connected then

$$C^*(-;\bar{\mathbb{F}}_p)\colon \operatorname{Ho}_p \mathcal{S}(X,Y) \to \operatorname{Ho}\mathcal{E}_{\bar{\mathbb{F}}_p}(C^*(Y;\bar{\mathbb{F}}_p), C^*(X;\bar{\mathbb{F}}_p))$$

is a bijection.

(This is slightly weaker than the theorem stated at the start of the lecture.)

Suffices to show that $Y \to \bar{U}_\bullet A$ is a p-adic equivalence for a cofibrant approximation $A \to C^*(Y;\bar{\mathbb{F}}_p)$.

True for $K(\mathbb{F}_p, n)$

Eilenberg–Moore: Homotopy pullbacks of finite type simply connected spaces go to homotopy pullback of E_∞ algebras.

For $K(\mathbb{Z}/p^k, n)$ use fibration seq. $K(\mathbb{Z}/p^k) \to K(\mathbb{Z}/p^{k-1}) \to K(\mathbb{Z}/p, n+1)$. $K(\mathbb{Z},n)^\wedge_p \simeq K(\mathbb{Z}^\wedge_p, n) \simeq \operatorname{holim} K(\mathbb{Z}/p^k, n)$. Follows for all finite type $K(\pi, n)$.

Induction up principal Postnikov tower.

The Unit for E_∞ $\mathbb{F}_p$-algebras

So what does

$$\operatorname{Ho}_p \mathcal{S}(X,Y) \to Ho\mathcal{E}_{\mathbb{F}_p}(C^*(Y;\mathbb{F}_p), C^*(X;\mathbb{F}_p)) \simeq \operatorname{Ho}_p \mathcal{S}(X, RU_\bullet C^*(Y;\mathbb{F}_p))$$

look like?

Answer: $Y \to \Lambda Y^\wedge_p$

$C^*(Y;\bar{\mathbb{F}}_p) \simeq C^*(Y;\mathbb{F}_p) \otimes \bar{\mathbb{F}}_p$

So if $A \to C^*(Y;\mathbb{F}_p)$ is a cofibrant E_∞ $\mathbb{F}_p$-algebra approximation, then $A \otimes \bar{\mathbb{F}}_p \to C^*(Y;\bar{\mathbb{F}}_p)$ is a cofibrant E_∞ $\mathbb{F}_p$-algebra approximation.

Then $Y \simeq \bar{U}_\bullet(A \otimes \bar{\mathbb{F}}_p) = \mathcal{E}_{\bar{\mathbb{F}}_p}(A \otimes \bar{\mathbb{F}}_p, C^*(\Delta^\bullet;\bar{\mathbb{F}}_p)) \cong \mathcal{E}_{\mathbb{F}_p}(A, C^*(\Delta^\bullet;\bar{\mathbb{F}}_p))$

(Assuming without loss of generality that Y is p-complete)

$\mathbb{F}_p$ is (homotopy) fixed points of Frobenius, $U_\bullet A$ becomes homotopy fixed points of an automorphism Φ on $\bar{U}_\bullet(A \otimes \bar{\mathbb{F}}_p)$. The weak equivalence $Y \to \bar{U}_\bullet(A \otimes \bar{\mathbb{F}}_p)$ factors through the fixed points of this automorphism, so we get weak equivalences

$$U_\bullet A \simeq (\bar{U}_\bullet(A \otimes \bar{\mathbb{F}}_p))^{h\Phi} \simeq Y^{h\Phi} = \Lambda Y$$