Localization Sequences in *THH*

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Banff Workshop
on
Algebraic K-Theory and Equivariant Homotopy Theory

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Overview

Localization Sequences in *THH*

- Joint work with Andrew Blumberg

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Localization Sequences in *THH* and *TC*

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Localization Sequences in *THH* and *TC* II

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2nd Goal: Understand the relationship to already known localization sequences in *THH* and *TC*

3rd Goal: Make sense of the *THH* of Waldhausen categories

Quillen Localization Sequence

Let R be a discrete valuation ring, with residue field k and field of fractions F. Then there is a cofibration sequence of K-theory spectra

$$K(k) \to K(R) \to K(F) \to$$

R local ring, PID
unique (non-zero) irreducible elt π

$k = R/\pi$
$F = R[\pi^{-1}]$

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Secretly $K(k)$ is really the K-theory of the category of finitely generated torsion R-modules.

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Hesselholt–Madsen Theorem

Let R be a complete discrete valuation ring, with residue field k perfect of characteristic $p > 2$ and field of fractions F of characteristic zero, containing the p^n-th roots of unity. Then:

- $K(F; \mathbb{Z}/p^n)$ can be computed in terms of the De Rham–Witt complex.
- F satisfies the Lichtenbaum-Quillen conjecture.

$p=2$ Voevodsky

Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.

Argument

1. McCarthy Theorem: If $A \to B$ is surjective with nilpotent kernel

$$\begin{array}{ccc} K(A)_p^\wedge & \longrightarrow & K(B)_p^\wedge \\ \downarrow & & \downarrow \\ TC(A)_p^\wedge & \longrightarrow & TC(B)_p^\wedge \end{array}$$

   is homotopy cartesian.

2. Quillen–Krasner Theorem: If k is a perfect field of characteristic p, then $K(k)_p^\wedge \simeq H\mathbb{Z}_p^\wedge$.

3. Suslin results imply $K(R)_p^\wedge \simeq \operatorname{holim} K(R/\pi^n)_p^\wedge$.

Hesselholt–Madsen then show:

$$K(k)_p^\wedge \simeq TC(k)[0,\infty) \qquad K(R)_p^\wedge \simeq TC(R)[0,\infty)$$

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Key step to enable computation

Identify the cofiber of $TC(k) \to TC(R)$ in intrinsic terms.

Identify the cofiber of $THH(k) \to THH(R)$ in intrinsic terms.

Note: Cofiber is not $THH(F)$

Example

$R = \mathbb{Z}_p^\wedge, \qquad k = \mathbb{Z}/p, \qquad F = \mathbb{Q}_p^\wedge$

- $THH(\mathbb{Z}/p) = \bigvee \Sigma^{2n} H\mathbb{Z}/p$
- $THH(\mathbb{Z}_p^\wedge) = \bigvee \Sigma^{2n-1} H\mathbb{Z}_p^\wedge/n$
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Idea

Use Waldhausen's localization sequence in K-theory to construct a localization sequence in THH.

$$THH(k) \to THH(R) \to THH(R \mid F) \to$$

Recall

$$THH(k) \cong THH(\mathcal{T}) = N^{cyc}_{\text{Bök}}(S_\bullet \mathcal{T}) \simeq N^{cyc}_{\text{Bök}}(S_\bullet N^i_\bullet \mathcal{T})$$
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$\mathcal{T}$ = Torsion f.g. R-modules
$\mathcal{M}$ = All f.g. R-modules
$\mathcal{C}$ = Complexes of f.g. R-modules, $\mathcal{C}^q$ = q-acyclic complexes
w = weak equivalences = quasi-isomorphisms
q = mod torsion equivalences

$N^{cyc}_{\text{Bök}}(S_\bullet -)$ is the THH of an exact category functor.

Construction

$$\begin{array}{ccccc} THH(k) & \longrightarrow & THH(R) & \longrightarrow & THH(R \mid F) \longrightarrow \\ \| & & \| & & \\ N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^q) & & N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}) & & N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C}) \end{array}$$

Definition

$THH(R \mid F) = N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C})$

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Waldhausen Square

The square

$$\begin{tikzcd} (S_\bullet N^w_\bullet \mathcal{C}^q) \arrow[r] \arrow[d] & (S_\bullet N^q_\bullet \mathcal{C}^q) \arrow[d] \\ (S_\bullet N^w_\bullet \mathcal{C}) \arrow[r] & (S_\bullet N^q_\bullet \mathcal{C}) \end{tikzcd}$$

$\simeq *$

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Theorem (Waldhausen/McCarthy)

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is homotopy cartesian

Ausoni–Rognes Computations

Ausoni and Ausoni–Rognes compute

- $K(ku)$ and $K(\ell_p^\wedge)$
- $K(KU)$ and $K(L_p^\wedge)$

mod p and v_1 assuming:

- Localization sequence in K-theory

$$K(H\mathbb{Z}) \to K(ku) \to K(KU) \to$$
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The localization sequence for $K(KU)$

Use Waldhausen's square again:

Theorem (Waldhausen)

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But now:
$\mathcal{C}$ = finite cell ku-modules
w = weak equivalences
q = maps that become weak equivalences after inverting Bott element

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First localization sequence for *THH*

Categories $N_m^w \mathcal{C}$ not exact categories, but are *spectral categories*

$\mathcal{C}$ = cell ku-modules

Use natural mapping spectra in $\mathcal{C}$.

Get mapping spectra for diagram categories

$$N_m^w \mathcal{C}, \qquad S_n N_m^w \mathcal{C}, \qquad S_n N_m^w \mathcal{C}^q, \qquad \text{etc.}$$

Then apply $N_{\text{Bök}}^{cyc}$ to square

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First localization sequence for *THH*

Categories $N_m^w\mathcal{C}$ not exact categories,
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Use natural mapping spectra in $\mathcal{C}$.

Get mapping spectra for diagram categories

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$$THH(N^w_\bullet \mathcal{C}^q) \to THH(ku) \to THH(N^q_\bullet \mathcal{C}) \to$$

But $THH(N^q_\bullet \mathcal{C}) \simeq THH(KU)$
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How does this localization sequence fit in?

Quillen's cofiber sequence

$$K(k) \to K(R) \to K(F) \to$$

generalizes to Thomason–Trobaugh's cofiber sequence

$$K^B(X \text{ on } Y) \to K^B(X) \to K^B(X - Y) \to$$

"on" means "supported on"

Using *THH* of spectral categories of perfect complexes, get a cofiber sequence

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Localization sequence for *THH* of schemes

When $X = \operatorname{Spec} R$ for any commutative ring R,
we can use $\mathcal{C} = \mathcal{C}_{HR}$ finite cell HR-modules.

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$$THH(N_\bullet^w \mathcal{C}_{HR}^q) \to THH(N_\bullet^w \mathcal{C}_{HR}) \to THH(N_\bullet^q \mathcal{C}_{HR}) \to$$

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Why Thomason–Trobaugh sequence different from the Hesselholt-Madsen sequence?

Hesselholt–Madsen Sequence

- Treat category of complexes as an *exact* category
- Mapping spectra always Eilenberg–Mac Lane spectrum – no homotopy groups except in degree zero
- Meaning of mapping spectra?

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Mapping Sets vs. Mapping Spaces

Remarks on mapping sets in Hesselholt-Madsen construction

- Could have used just non-negative chain complexes instead of integer graded chain complexes
- Could have used simplicial modules instead of non-negative chain complexes
- Could have used mapping spaces (simplicial sets) instead of mapping sets

$$N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}) \simeq N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}_\bullet)$$
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Waldhausen Category

- Notion of weak equivalence
- Notion of cofibration
- Pushouts over cofibration (including sums)
- Nice relationship between weak equivalence and cofibration

Now also assume simplicial enrichment for mapping *spaces*

Without loss of generality: Modern homotopy theory says that this structure always arises and plays nicely with cofibrations and weak equivalences.

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From mapping space $\mathcal{C}(X, Y)$ get connective spectrum from gamma space

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Use this spectral enrichment to construct a new *THH*.

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Connective and non-connective enrichments

Connective enrichment vs. non-connective enrichment

$$\mathcal{C}(X,Y)^{\Gamma}(n) = |\mathcal{C}(X, \bigvee_{S^n_\bullet} Y)| \qquad \mathcal{C}(X,Y)^{S}(n) = |\mathcal{C}(X, Y \otimes S^n)|$$

Canonical map

$$\mathcal{C}(X,Y)^{\Gamma} \to \mathcal{C}(X,Y)^{S}$$

often connective cover,
e.g., when $\mathcal{C}(X,Y) \to \mathcal{C}(\Sigma X, \Sigma Y)$ is a weak equivalence

Trace map

$$\begin{tikzcd}
K(\mathcal{C}) = \mathrm{Ob}(S_\bullet N^w_\bullet \mathcal{C}) \arrow[r] \arrow[dr] & N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^{\Gamma}) \simeq W^{\Gamma} THH(\mathcal{C}) \arrow[d] \\
& N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^{S})
\end{tikzcd}$$

Do we actually get something different?

Let $\mathcal{E}$ be an exact category, viewed as a Waldhausen category with weak equivalences the isomorphisms and mapping spaces discrete.

Then $W^{\Gamma} THH(\mathcal{E})$ is the Dundas–McCarthy $THH(\mathcal{E})$.

Now let $\mathcal{E}$ be the exact category of locally free sheaves on a quasi-projective variety X.

Can give $\mathcal{E}$ a non-connective spectra enrichment $\mathcal{E}^S$ that correctly captures the fact that Ext in $\mathcal{E}$ can be non-trivial.

When X is affine $\mathcal{E} \simeq \mathcal{E}^S$ and $THH(\mathcal{E}) \simeq THH(\mathcal{E}^S)$.

Using $\mathcal{E}^S$, $\pi_* THH(\mathcal{E}^S)$ is a quasi-coherent sheaf over X. This does not hold in general for $\pi_* THH(\mathcal{E})$.

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What about for categories of R-modules in spectra

Theorem (Sphere Theorem)

Let $\mathcal{C}_R$ be the Waldhausen category of finite cell R-modules for R a connective EKMM S-algebra or R a simplicial ~~R-algebra~~ ring. Then $W^{\Gamma} THH(\mathcal{C}_R) \simeq THH(R)$.

Does not hold if we do not assume connective.

Theorem generalizes to any spectrally enriched Waldhausen category $\mathcal{C}$ that has a set Q of generators such that the natural mapping spectra between objects in Q are connective.

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$S_* \mathcal{C}^{\Gamma}$

$S_* N^q_* \mathcal{C}^{\Gamma}$

A devissage theorem

Theorem

Let R be a connective EKMM S-algebra with $\pi_0 R$ connective and let $\mathcal{P}$ be the category of cell R-algebras that have finitely many non-zero homotopy groups all of which are finitely generated. Then $W^\Gamma THH(\mathcal{P}) \simeq THH(\pi_0 R)$.

In particular $W^\Gamma THH(\mathcal{P})$ has zero negative homotopy groups. Using the natural (non-connective) mapping spectra, usually get negative homotopy groups for $THH(\mathcal{P})$ unless $R \simeq H\pi_0 R$.

For $R = ku$, $\mathcal{P} \simeq \mathcal{C}^q$

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Corollary: Localization sequence for the *THH* of topological K-theory

Define $W^\Gamma THH(ku \mid KU)$ as $N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C}^\Gamma)$

$\mathcal{C}$ = cell ku-modules

Corollary

The cofiber sequence

$$N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet(\mathcal{C}^\Gamma)^q) \to N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^\Gamma) \to N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C}^\Gamma) \to$$

is weakly equivalent to a cofiber sequence

$$THH(\mathbb{Z}) \to THH(ku) \to \boxed{W^\Gamma THH(ku \mid KU)} \to$$

where the map $THH(\mathbb{Z}) \to THH(ku)$ is a certain previously known transfer map.

Summary

For $\mathcal{C}$ the category of cell ku-modules

Connective and non-connective spectral enrichments gives two different localization sequences:

$$\begin{tikzcd}
K(H\mathbb{Z}) \arrow[r] \arrow[d] & K(ku) \arrow[r] \arrow[d] & K(KU) \arrow[r] \arrow[d] & {} \\
THH(H\mathbb{Z}) \arrow[r] & THH(ku) \arrow[r] & W^{\Gamma} THH(ku \mid KU) \arrow[r] & {}
\end{tikzcd}$$

$THH(ku \text{ on } H\mathbb{Z})$ $\quad$ $THH(ku)$ $\quad$ $THH(KU)$

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S. N. C