Localization Sequences in $THH$

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Banff Workshop on
Algebraic $K$-Theory and Equivariant Homotopy Theory

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Overview

Localization Sequences in $THH$

- Joint work with Andrew Blumberg
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Localization Sequences in $THH$ and $TC$

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Localization Sequences in $THH$ and $TC$ II

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Localization Sequences in \( THH \) and \( TC \) II

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**Main Goal**: Explain Ausoni–Rognes/Hesselholt conjecture about the localization sequences for $THH(ku)$ (and $TC(ku)$)
Localization Sequences in $THH$ and $TC$ II

- Joint work with Andrew Blumberg

**Main Goal**: Explain Ausoni–Rognes/Hesselholt conjecture about the localization sequences for $THH(ku)$ (and $TC(ku)$)

**2nd Goal**: Understand the relationship to already known localization sequences in $THH$ and $TC$
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Localization Sequences in $THH$ and $TC$ II

- Joint work with Andrew Blumberg

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**2nd Goal**: Understand the relationship to already known localization sequences in $THH$ and $TC$

**3rd Goal**: Make sense of the $THH$ of Waldhausen categories
Let $R$ be a discrete valuation ring, with residue field $k$ and field of fractions $F$. Then there is a cofibration sequence of $K$-theory spectra

$$K(k) \to K(R) \to K(F) \to$$

This uses the $K$-theory of abelian categories. Secretly $K(k)$ is really the $K$-theory of the category of finitely generated torsion $R$-modules. (Devissage theorem.)

$R$ is a local ring, PID unique (non-zero) irreducible elt $\pi$

$k = R/\pi$

$F = R[\pi^{-1}]$
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Quillen Localization Sequence

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(Devissage theorem.)
Let $R$ be a complete discrete valuation ring, with residue field $k$ perfect of characteristic $p > 2$ and field of fractions $F$ of characteristic zero, containing the $p^n$-th roots of unity. Then:

- $K(F; \mathbb{Z}/p^n)$ can be computed in terms of the De Rham–Witt complex.
- $F$ satisfies the Lichtenbaum-Quillen conjecture.

Argument

1. McCarthy Theorem: If $A \to B$ is surjective with nilpotent kernel

\[
\begin{array}{c c c}
K(A)_{\wedge}^p & \longrightarrow & K(B)_{\wedge}^p \\
\downarrow & & \downarrow \\
TC(A)_{\wedge}^p & \longrightarrow & TC(B)_{\wedge}^p
\end{array}
\]

is homotopy cartesian.

2. Quillen–Krasner Theorem: If $k$ is a perfect field of characteristic $p$, then $K(k)_{\wedge}^p \cong H\mathbb{Z}_{\wedge}^p$.

3. Suslin results imply $K(R)_{\wedge}^p \cong \operatorname{holim} K(R/\pi^n)_{\wedge}^p$.

Hesselholt–Madsen then show:

\[
K(k)_{\wedge}^p \cong TC(k)[0, \infty), \quad K(R)_{\wedge}^p \cong TC(R)[0, \infty).
\]
1 McCarthy Theorem: If $A \to B$ is surjective with nilpotent kernel

$$K(A)_p^\wedge \longrightarrow K(B)_p^\wedge$$

$$\downarrow \quad \downarrow$$

$$TC(A)_p^\wedge \longrightarrow TC(B)_p^\wedge$$

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2 Quillen–Krasner Theorem: If $k$ is a perfect field of characteristic $p$, then $K(k)_p^\wedge \simeq H\mathbb{Z}_p^\wedge$.

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1 McCarthy Theorem: If $A \to B$ is surjective with nilpotent kernel

$$
\begin{array}{c}
K(A)^\wedge_p \longrightarrow K(B)^\wedge_p \\
\downarrow \hspace{0.5cm} \downarrow \\
TC(A)^\wedge_p \longrightarrow TC(B)^\wedge_p
\end{array}
$$

is homotopy cartesian.

2 Quillen–Krasner Theorem: If $k$ is a perfect field of characteristic $p$, then $K(k)^\wedge_p \simeq H\mathbb{Z}_p^\wedge$.

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$$
McCarthy Theorem: If \( A \to B \) is surjective with nilpotent kernel

\[
\begin{align*}
K(A)_{\mathbb{Z}_p}^\wedge & \longrightarrow K(B)_{\mathbb{Z}_p}^\wedge \\
\downarrow & \quad \downarrow \\
TC(A)_{\mathbb{Z}_p}^\wedge & \longrightarrow TC(B)_{\mathbb{Z}_p}^\wedge
\end{align*}
\]

is homotopy cartesian.

Quillen–Krasner Theorem: If \( k \) is a perfect field of characteristic \( p \), then \( K(k)_{\mathbb{Z}_p}^\wedge \simeq H\mathbb{F}_p^\wedge \).

Suslin results imply \( K(R)_{\mathbb{Z}_p}^\wedge \simeq \text{holim} K(R/\pi^n)_{\mathbb{Z}_p}^\wedge \).

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K(k)^\wedge_p & \simeq TC(k)[0, \infty) \\
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\end{align*}
\]
Key step to enable computation

Identify the cofiber of $\text{TC}(k) \to \text{TC}(R)$ in intrinsic terms.

Identify the cofiber of $\text{THH}(k) \to \text{THH}(R)$ in intrinsic terms.

Note: Cofiber is not $\text{THH}(F)$

Example

<table>
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<tr>
<th>$R = \mathbb{Z}^\wedge_p$,</th>
<th>$k = \mathbb{Z}/p$,</th>
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*This step is not the “big idea” in the paper. Big idea is the computation itself and interpretation in terms of De Rham–Witt. This is just the piece of theory that makes it go.*
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Identify the cofiber of $TC(k) \to TC(R)$ in intrinsic terms.

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Idea

Use Waldhausen’s localization sequence in $K$-theory to construct a localization sequence in $THH$.

\[ THH(k) \to THH(R) \to THH(R \mid F) \to \]

Recall

\[ THH(k) \cong THH(T) = N_{\text{cyc}}^{\text{Bök}}(S \cdot T) \cong N_{\text{cyc}}^{\text{Bök}}(S \cdot N^i \cdot T) \]

\[ THH(R) \cong THH(M) = N_{\text{cyc}}^{\text{Bök}}(S \cdot M) \cong N_{\text{cyc}}^{\text{Bök}}(S \cdot N^i \cdot M) \]

$T = \text{Torsion f.g. } R\text{-modules}$

$M = \text{All f.g. } R\text{-modules}$

$N_{\text{cyc}}^{\text{Bök}}(S \cdot -)$ is the $THH$ of an exact category functor.
Use Waldhausen’s localization sequence in $K$-theory to construct a localization sequence in $\text{THH}$.

$$\text{THH}(k) \rightarrow \text{THH}(R) \rightarrow \text{THH}(R \mid F) \rightarrow$$

Recall

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$$\text{THH}(R) \cong \text{THH}(\mathcal{M}) = N_{\text{Bök}}^{\text{cyc}}(S \cdot \mathcal{M}) \cong N_{\text{Bök}}^{\text{cyc}}(S \cdot N^i \mathcal{M})$$

$\mathcal{T} =$ Torsion f.g. $R$-modules

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$\mathcal{T} = \text{Torsion f.g. } R\text{-modules}$

$\mathcal{M} = \text{All f.g. } R\text{-modules}$

$\mathcal{C} = \text{Complexes of f.g. } R\text{-modules}$

$w = \text{weak equivalences } = \text{quasi-isomorphisms}$

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$$THH(R) \cong THH(\mathcal{M}) = \text{N}^{\text{cyc}}_{\text{Bök}}(S \cdot \mathcal{M}) \cong \text{N}^{\text{cyc}}_{\text{Bök}}(S \cdot N^i \mathcal{M})$$

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$\mathcal{T} = \text{Torsion f.g. } R\text{-modules}$

$\mathcal{M} = \text{All f.g. } R\text{-modules}$

$\mathcal{C} = \text{Complexes of f.g. } R\text{-modules}$, $\mathcal{C}^q = q\text{-acyclic complexes}$

$w = \text{weak equivalences = quasi-isomorphisms}$

$q = \text{mod torsion equivalences}$

$N^\text{cyc}_{\text{Bök}}(S \cdot -)$ is the $THH$ of an exact category functor.
Construction

\[
\text{THH}(k) \quad \lll \quad \text{THH}(R) \quad \lll \quad \text{THH}(R \mid F)
\]

\[
\text{N}_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C^q) \quad \lll \quad \text{N}_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C) \quad \lll \quad \text{N}_{\text{Bök}}^{\text{cyc}}(S \cdot N^q C)
\]

Definition

\[
\text{THH}(R \mid F) = \text{N}_{\text{Bök}}^{\text{cyc}}(S \cdot N^q C)
\]
Construction

\[ \text{THH}(k) \rightarrow \text{THH}(R) \rightarrow \text{THH}(R \mid F) \]

\[ \text{N}_{\text{Bök}}^{\text{cyc}}(S \cdot N^w \cdot C^q) \quad \text{N}_{\text{Bök}}^{\text{cyc}}(S \cdot N^w \cdot C) \quad \text{N}_{\text{Bök}}^{\text{cyc}}(S \cdot N^q \cdot C) \]

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\[ \text{N}^\text{cyc}_{\text{Bök}}(S \cdot N^w C^q) \quad \text{N}^\text{cyc}_{\text{Bök}}(S \cdot N^w C) \quad \text{N}^\text{cyc}_{\text{Bök}}(S \cdot N^q C) \]

Waldhausen Square

The square

\[ (S \cdot N^w C^q) \quad (S \cdot N^q C^q) \]

\[ (S \cdot N^w C) \quad (S \cdot N^q C) \]
Construction

The square

\[ \begin{array}{ccc}
\text{THH}(k) & \xrightarrow{\sim} & \text{THH}(R) \\
N_{\text{Böck}}^{\text{cyc}}(S \cdot N^w C^q) & \xrightarrow{\sim} & N_{\text{Böck}}^{\text{cyc}}(S \cdot N^w C) \\
\end{array} \]

\[ \begin{array}{ccc}
\text{THH}(R \mid F) & \xrightarrow{\sim} & \text{THH}(k) \\
N_{\text{Böck}}^{\text{cyc}}(S \cdot N^q C) & \xrightarrow{\sim} & N_{\text{Böck}}^{\text{cyc}}(S \cdot N^w C) \\
\end{array} \]

is homotopy cartesian

Theorem (Waldhausen/McCarthy)

The square

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N_{\text{Böck}}^{\text{cyc}}(S \cdot N^w C^q) & \xrightarrow{\sim} & N_{\text{Böck}}^{\text{cyc}}(S \cdot N^q C^q) \\
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is homotopy cartesian
Ausoni–Rognes Computations

Ausoni and Ausoni–Rognes compute

- $K(ku)$ and $K(\ell_p^\wedge)$
- $K(KU)$ and $K(L_p^\wedge)$

mod $p$ and $v_1$ assuming:

- Localization sequence in $K$-theory

\[
K(H\mathbb{Z}) \to K(ku) \to K(KU) \to K(H\mathbb{Z}_p^\wedge) \to K(\ell_p^\wedge) \to K(L_p^\wedge) \to
\]

- Localization sequence in $THH$

\[
THH(H\mathbb{Z}) \to THH(ku) \to THH(KU) \to THH(H\mathbb{Z}_p^\wedge) \to THH(\ell_p^\wedge) \to THH(L_p^\wedge) \to
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\]
The localization sequence for $K(KU)$

Use Waldhausen’s square again:

Theorem (Waldhausen)

The square

$$
\begin{array}{ccc}
\text{Ob}(S \cdot N^w C^q) & \rightarrow & \text{Ob}(S \cdot N^q C^q) \\
\downarrow & & \downarrow \\
\text{Ob}(S \cdot N^w C) & \rightarrow & \text{Ob}(S \cdot N^q C)
\end{array}
$$

is homotopy cartesian

But now:

$\mathcal{C} =$ finite cell $ku$-modules

$w =$ weak equivalences

$q =$ maps that become weak equivalences after inverting Bott element

$\text{Ob } S_\bullet(\_)$ is the $K$-theory of a Waldhausen category
The localization sequence for $K(KU)$

Use Waldhausen’s square again:

**Theorem (Waldhausen)**

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\[
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But now:

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$\text{Ob}(S \cdot N^w C^q) \rightarrow \text{Ob}(S \cdot N^q C^q)$

$\text{Ob}(S \cdot N^w C) \rightarrow \text{Ob}(S \cdot N^q C)$

is homotopy cartesian

But now:

$C = \text{finite cell } ku\text{-modules}$

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Devissage theorem identifying $\text{Ob}(S \cdot N^w C^q)$ as $K(H\mathbb{Z})$
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Devissage theorem identifying $\text{Ob}(S \cdot N^w C^q)$ as $K(H\mathbb{Z})$
First localization sequence for \( THH \)

Categories \( N_m^w C \) not exact categories, but are *spectral categories*

Use natural mapping spectra in \( C \).

Get mapping spectra for diagram categories

\[
N_m^w C, \quad S_n N_m^w C, \quad S_n N_m^w C^q, \quad \text{etc.}
\]

Then apply \( N_{\text{Bök}}^{\text{cyc}} \) to square

\[
\begin{align*}
S \cdot N^w C^q & \rightarrow S \cdot N^q C^q \\
\downarrow & \\
S \cdot N^w C & \rightarrow S \cdot N^q C
\end{align*}
\]
First localization sequence for \( THH \)

Categories \( N^w_mC \) not exact categories, but are \textit{spectral categories}

Use natural mapping spectra in \( C \).

Get mapping spectra for diagram categories

\[
N^w_mC, \quad S_nN^w_mC, \quad S_nN^w_mC^q, \quad \text{etc.}
\]

Then apply \( N^\text{cyc}_{\text{B"ok}} \) to square

\[
\begin{array}{ccc}
S \cdot N^w_mC^q & \rightarrow & S \cdot N^q_mC^q \\
\downarrow & & \downarrow \\
S \cdot N^w_mC & \rightarrow & S \cdot N^q_mC
\end{array}
\]
First localization sequence for $THH$

Categories $N^w_m C$ not exact categories, but are *spectral* categories

Use natural mapping spectra in $C$.

Get mapping spectra for diagram categories

$N^w_m C$, $S_n N^w_m C$, $S_n N^w_m C^q$, etc.

Then apply $N^{cyc}_{Bök}$ to square

$$
\begin{align*}
S \cdot N^w C & \rightarrow S \cdot N^q C^q \\
\downarrow & \\
S \cdot N^w C & \rightarrow S \cdot N^q C
\end{align*}
$$
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\[N^w_mC, \quad S_nN^w_mC, \quad S_nN^w_MC^q, \quad \text{etc.}\]

Then apply \(N^\text{cyc}_{\text{Bök}}\) to square

\[
\begin{array}{ccc}
S\cdot N^w_MC^q & \rightarrow & S\cdot N^q_NC^q \\
\downarrow & & \downarrow \\
S\cdot N^w_C & \rightarrow & S\cdot N^q_C
\end{array}
\]
First localization sequence for \( \text{THH} \)

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Get mapping spectra for diagram categories

\[
N^w_m C, \quad S_n N^w_m C, \quad S_n N^w_m C^q, \quad \text{etc.}
\]

Then apply \( N^\text{cyc} \) to square

\[
S \bullet N^w C^q \rightarrow S \bullet N^q C^q \\
\downarrow \quad \downarrow \\
S \bullet N^w C \quad ightarrow \quad S \bullet N^q C
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$$\downarrow \quad \downarrow$$

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First localization sequence for \textit{THH}

Categories $N^w_m \mathcal{C}$ not exact categories, but are \textit{spectral categories}

Use natural mapping spectra in $\mathcal{C}$.

Get mapping spectra for diagram categories

$N^w_m \mathcal{C}$, $S_n N^w_m \mathcal{C}$, $S_n N^w_m \mathcal{C}^q$, etc.

Then apply $N^\text{cyc}_{\text{Bök}}$ to square

\[
\begin{array}{ccc}
S_n N^w_m \mathcal{C} & \rightarrow & S_n N^q \mathcal{C} \\
\downarrow & & \downarrow \\
S_n N^w_m \mathcal{C} & \rightarrow & S_n N^q \mathcal{C}
\end{array}
\]
First localization sequence for \( THH \) II

Get homotopy (co)cartesian square

\[
\begin{align*}
N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C^q) & \rightarrow N_{\text{Bök}}^{\text{cyc}}(S \cdot N^q C^q) \\
\downarrow & \\
N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C) & \rightarrow N_{\text{Bök}}^{\text{cyc}}(S \cdot N^q C)
\end{align*}
\]

and cofiber sequence

\[
THH(N^w C^q) \rightarrow THH(ku) \rightarrow THH(N^q C) \rightarrow
\]

But \( THH(N^q C) \simeq THH(KU) \)
and \( THH(N^q C^q) \not\simeq THH(H\mathbb{Z}) \)
First localization sequence for $THH$ II

Get homotopy (co)cartesian square

$\mathcal{N}^{\text{cyc}}_{\text{B"ok}}(S \cdot N_w^q C_q) \rightarrow \mathcal{N}^{\text{cyc}}_{\text{B"ok}}(S \cdot N_q^q C_q)$

$\downarrow \downarrow \downarrow$

$\mathcal{N}^{\text{cyc}}_{\text{B"ok}}(S \cdot N_w^w C) \rightarrow \mathcal{N}^{\text{cyc}}_{\text{B"ok}}(S \cdot N_q^q C)$

and cofiber sequence

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\[
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\downarrow & \\
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How does this localization sequence fit in?

Quillen’s cofiber sequence

\[ K(k) \to K(R) \to K(F) \to \]

generalizes to Thomason–Trobaugh’s cofiber sequence

\[ K^B(X \text{ on } Y) \to K^B(X) \to K^B(X - Y) \to \]

“on” means “supported on”

Using \( THH \) of spectral categories of perfect complexes, get a cofiber sequence

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compatible with trace map from Thomason–Trobaugh cofiber sequence.
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Localization sequence for \( THH \) of schemes

When \( X = \text{Spec} \, R \) for any commutative ring \( R \), we can use \( C = C_{HR} \) finite cell \( HR \)-modules.

Then cofiber sequence

\[
\text{THH}(N_w^r C_{HR}^q) \to \text{THH}(N_w^r C_{HR}) \to \text{THH}(N_q^r C_{HR}) \to
\]

is equivalent to localization sequence for \( THH \) of schemes

\[
\text{THH}(X \text{ on } Y) \to \text{THH}(X) \to \text{THH}(X - Y) \to
\]

For example, for \( R \) a discrete valuation ring, with residue field \( k \) and field of fractions \( F \), this is a sequence

\[
\text{THH}(\text{Spec} \, R \text{ on } \text{Spec} \, k) \to \text{THH}(R) \to \text{THH}(F) \to
\]

Because of this connection, will call this \( THH \) sequence the “Thomason–Trobaugh” sequence to distinguish from the other “Hesselholt–Madsen” sequence.
Localization sequence for $THH$ of schemes

When $X = \text{Spec } R$ for any commutative ring $R$, we can use $C = C_{HR}$ finite cell $HR$-modules.

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is equivalent to localization sequence for $THH$ of schemes

$$THH(X \text{ on } Y) \to THH(X) \to THH(X - Y) \to$$

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\[
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Why Thomason–Trobaugh sequence different from the Hesselholt-Madsen sequence?

**Hesselholt–Madsen Sequence**
- Treat category of complexes as an *exact* category
- Mapping spectra always Eilenberg–Mac Lane spectrum – no homotopy groups except in degree zero
- Meaning of mapping spectra?

**Thomason–Trobaugh Sequence**
- Treat category of modules as a *spectral* category
- Mapping spectra generally have both positive and negative homotopy groups
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Mapping Sets vs. Mapping Spaces

Remarks on mapping sets in Hesselholt-Madsen construction

- Could have used just non-negative chain complexes instead of integer graded chain complexes
- Could have used simplicial modules instead of non-negative chain complexes
- Could have used mapping spaces (simplicial sets) instead of mapping sets

\[
N_{\text{B{"o}k}}^{\text{cyc}}(S \cdot N^w \mathcal{C}) \cong N_{\text{B{"o}k}}^{\text{cyc}}(S \cdot N^w \mathcal{C}) \\
\cong N_{\text{B{"o}k}}^{\text{cyc}}(S \cdot \bar{N}^w \mathcal{C}) \cong N_{\text{B{"o}k}}^{\text{cyc}}(S \cdot \mathcal{C})
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$$\cong N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w \cdot \overline{C}) \cong N_{\text{Bök}}^{\text{cyc}}(S \cdot \overline{C})$$
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\[
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\]
Simplicially enriched Waldhausen categories

Waldhausen Category

- Notion of weak equivalence
- Notion of cofibration
- Pushouts over cofibration (including sums)
- Nice relationship between weak equivalence and cofibration

Now also assume simplicial enrichment for mapping spaces

Without loss of generality: Modern homotopy theory says that this structure always arises and plays nicely with cofibrations and weak equivalences.
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Connective spectral enrichment

From mapping space $\mathcal{C}(X, Y)$ get connective spectrum from gamma space

$$\mathcal{C}(X, Y)_m^{\Gamma} = \mathcal{C}(X, \bigvee_{m} Y)$$

Use this spectral enrichment to construct a new $THH$.

**Definition**

$$W^{\Gamma} THH(\mathcal{C}) := N^{cyc}_{Bök}(S \mathcal{C}^{\Gamma})$$
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Trace Map

Definition

\[ W^\Gamma \text{THH}(\mathcal{C}) := N_{\text{Bök}}^{\text{cyc}}(S \cdot C^\Gamma) \]

Again \( N_{\text{Bök}}^{\text{cyc}}(S \cdot C^\Gamma) \cong N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C^\Gamma) \)

Get trace map

\[ K(\mathcal{C}) = \text{Ob}(S \cdot N^w C) \to N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C^\Gamma) \cong W^\Gamma \text{THH}(\mathcal{C}) \]

as inclusion of objects

When \( \mathcal{C} \) has intrinsic mapping spectra, trace map factors through this non-connective enrichment.
Trace Map

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Connective and non-connective enrichments

Connective enrichment vs. non-connective enrichment

\[ C(X, Y)^\Gamma(n) = |C(X, \bigvee_{S^n} Y)| \quad C(X, Y)^S(n) = |C(X, Y \otimes S^n)| \]

Canonical map

\[ C(X, Y)^\Gamma \to C(X, Y)^S \]

often connective cover, e.g., when \( C(X, Y) \to C(\Sigma X, \Sigma Y) \) is a weak equivalence

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\[ K(C) = \text{Ob}(S \cdot N_w C) \xrightarrow{N^{cyc}_{\text{Bök}}(S \cdot N_w C)} N^{cyc}_{\text{Bök}}(S \cdot N_w C) \cong W^\Gamma \text{THH}(C) \]
Let \( \mathcal{E} \) be an exact category, viewed as a Waldhausen category with weak equivalences the isomorphisms and mapping spaces discrete.

Then \( W^\Gamma THH(\mathcal{E}) \) is the Dundas–McCarthy \( THH(\mathcal{E}) \).

Now let \( \mathcal{E} \) be the exact category of locally free sheaves on a quasi-projective variety \( X \).

Can give \( \mathcal{E} \) a non-connective spectra enrichment \( \mathcal{E}^S \) that correctly captures the fact that Ext in \( \mathcal{E} \) can be non-trivial.

When \( X \) is affine \( \mathcal{E} \simeq \mathcal{E}^S \) and \( THH(\mathcal{E}) \simeq THH(\mathcal{E}^S) \).

Using \( \mathcal{E}^S \), \( \pi_* THH(\mathcal{E}^S) \) is a quasi-coherent sheaf over \( X \). This does not hold in general for \( \pi_* THH(\mathcal{E}) \).
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What about for categories of $R$-modules in spectra

**Theorem (Sphere Theorem)**

Let $\mathcal{C}_R$ be the Waldhausen category of finite cell $R$-modules for $R$ a connective EKMM $S$-algebra or $R$ a simplicial $R$-algebra. Then $W^\Gamma \text{THH}(\mathcal{C}_R) \simeq \text{THH}(R)$.

Does not hold if we do not assume connective.

Theorem generalizes to any spectrally enriched Waldhausen category $\mathcal{C}$ that has a set $Q$ of generators such that the natural mapping spectra between objects in $Q$ are connective.
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Theorem

Let $R$ be a connective EKMM $S$-algebra with $\pi_0 R$ connective and let $\mathcal{P}$ be the category of cell $R$-algebras that have finitely many non-zero homotopy groups all of which are finitely generated. Then $W^\Gamma THH(\mathcal{P}) \simeq THH(\pi_0 R)$.

In particular $W^\Gamma THH(\mathcal{P})$ has zero negative homotopy groups. Using the natural (non-connective) mapping spectra, usually get negative homotopy groups for $THH(\mathcal{P})$ unless $R \simeq H\pi_0 R$.

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Corollary: Localization sequence for the $THH$ of topological $K$-theory

Define $W^\Gamma THH(ku \mid KU)$ as $N^{cyc}_{Bök}(S \cdot N^q C^\Gamma)$

Corollary

The cofiber sequence

$$N^{cyc}_{Bök}(S \cdot N^w (C^\Gamma) q) \rightarrow N^{cyc}_{Bök}(S \cdot N^w C^\Gamma) \rightarrow N^{cyc}_{Bök}(S \cdot N^q C^\Gamma) \rightarrow$$

is weakly equivalent to a cofiber sequence

$$THH(\mathbb{Z}) \rightarrow THH(ku) \rightarrow W^\Gamma THH(ku \mid KU)$$

where the map $THH(\mathbb{Z}) \rightarrow THH(ku)$ is a certain previously known transfer map.
Summary

For $\mathcal{C}$ the category of cell $ku$-modules

Connective and non-connective spectral enrichments gives two different localization sequences:

\[
\begin{align*}
K(H\mathbb{Z}) & \to K(ku) \to K(KU) \\
\downarrow & \quad \downarrow & \quad \downarrow \\
THH(H\mathbb{Z}) & \to THH(ku) \to W^\Gamma THH(ku \mid KU) \\
\end{align*}
\]

$THH(ku \text{ on } H\mathbb{Z}) \quad THH(ku) \quad THH(KU)$
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