Large Scale Open Subsets of Configuration Spaces and the Foundations of Factorization Homology

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Topology Seminar

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Overview

Joint work with Andrew Blumberg

Goal: Construct norms for positive codimensional (closed) subgroups of compact Lie groups

Intuition is $N_H^G A \simeq \int_{G/H} A$

- Looks good for G/H finite index: $N_H^G A \simeq A^{(G/H)}$
- Looks good for $G = \mathbb{T} = S^1$, $H = e$: $N_e^{\mathbb{T}} A \simeq THH(A)$

Project Stages

- Foundations of factorization homology (nonequivariantly)
- Equivariant homotopy of smash power
- Equivariant homotopy of equivariant factorization homology

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Foundations of factorization homology

Need arguments that

- Avoid simplicial approximation (quasicategorical methods)
- Avoid local-to-global methods starting from a point or configuration

Construct covers of configuration spaces by "large scale" open subsets with intersection combinatorics and homotopy types mirroring a "Quillen Theorem A" comparison space

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Outline

1. Brief introduction to factorization homology
2. Construction of factorization homology
3. Large scale open subsets associated to partitions
4. The Quillen Theorem A argument
5. Some examples

What kind of thing is factorization homology?

$(\infty, 1)$-categories

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$(\infty, 1)$-categories

- Topological categories / continuous functors (e.g., cofibrant fibrant orthogonal spectra)
- Relative topological categories / continuous functors that preserve weak equivalences (e.g., all orthogonal spectra with usual weak equivalences)

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Manifold category

- Objects: Smooth d-manifolds with finitely many components
- Morphisms: Smooth embeddings
- Notation: $\mathscr{E}(M, N)$

Framings (!)

Nonunital Variant

Only allow maps that are surjective on π_0 ($\mathscr{E}(\emptyset, M)$ empty rather than 1-point)

Factorization homology

For a fixed "disk algebra" A, $\int_M A$ is a continuous functor of M from $\mathscr{E}$ to orthogonal spectra

$\int_M A$ is a relative continuous functor of (M, A) from $\mathscr{E} \times \mathscr{A}lg^{\flat}_{\mathcal{D}}$ to orthogonal spectra

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- Morphisms $\mathscr{D}(m, n) = \mathscr{E}(\mathbb{R}^d \times \{1, \ldots, m\}, \mathbb{R}^d \times \{1, \ldots, n\})$

Little disk category

Subcategory $\mathcal{D}$ of $\mathscr{D}$ with same objects and morphisms $\mathcal{D}(m, n)$ the subset of $\mathscr{D}(m, n)$ where

- Each $\mathbb{R}^d \times \{i\} \to \mathbb{R}^d \times \{j\}$ is an affine linear map, orthogonal up to scalar $(\mathbb{R}^d \rtimes (O(d) \times \mathbb{R}^+))$
- Images of unit disks lands in unit disks
- Images of open unit disks are disjoint

Inclusion of $\mathcal{D}$ in $\mathscr{D}$ is an equivalence

Disk algebra

A strict symmetric monoidal functor from $\mathcal{D}$ or $\mathscr{D}$ to orthogonal spectra

Factorization homology

$\int_M A$ is the (homotopy) left Kan extension of A along the inclusion of $\mathcal{D}$ (or $\mathscr{D}$) in $\mathscr{E}$

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Properties of factorization homology (and the norm)

Factorization homology

1. Symmetric monoidal in the manifold: $\int_{M \amalg N} A \simeq \int_M A \wedge \int_N A$
2. Symmetric monoidal in the algebra: $\int_M (A \wedge B) \simeq \int_M A \wedge \int_M B$
3. Bundle property: "$\int_B (\int_{F \times \mathbb{R}^c} A)$"$\simeq \int_E A$
4. Commutative algebra property: If A is a commutative algebra, $\int_M A \simeq A \otimes M$
5. Gluing property: "$(\int_M A) \wedge_{(\int_{L \times \mathbb{R}} A)} (\int_N A)$"$\simeq \int_{M \cup_L N} A$

The norm

1. If $H < G$ is finite index and $K < H$, then $N_K^G A \simeq (N_K^H A)^{(G/H)}$
2. $N_H^G (A \wedge B) \simeq N_H^G A \wedge N_H^G B$
3. For $K < H < G$, $N_H^G N_K^H \simeq N_K^G A$
4. Restricted to commutative H-algebras, N_H^G is the free functor to commutative G-algebras
5. (No analogue)

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2. $N_H^G (A \wedge B) \simeq N_H^G A \wedge N_H^G B$
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4. Restricted to commutative H-algebras, N_H^G is the free functor to commutative G-algebras
5. (No analogue)

Properties of factorization homology (and the norm)

Factorization homology

1. Symmetric monoidal in the manifold: $\int_{M \amalg N} A \simeq \int_M A \wedge \int_N A$
2. Symmetric monoidal in the algebra: $\int_M (A \wedge B) \simeq \int_M A \wedge \int_M B$
3. Bundle property: "$\int_B (\int_{F \times \mathbb{R}^c} A)$"$\simeq \int_E A$
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Construction of factorization homology

Factorization homology $\int_{(-)} A$ is (homotopy) left Kan extension of A along $\mathcal{D} \to \mathscr{E}$

Notation: Write $\mathcal{E}_M(m)$ for $\mathscr{E}(\mathbb{R}^d \times \{1, \ldots, m\}, M)$ (contravariant functor of $\mathscr{D}$ or $\mathcal{D}$)

Homotopy coend construction of homotopy left Kan extension

$$\int_M A = \mathrm{hocoend}_{\mathcal{D}}(\mathcal{E}_M(-)_+ \wedge A^{(-)})$$

Bar construction for homotopy coend $B(\mathcal{E}_M, \mathcal{D}, A)$

$$B_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \wedge A^{(k_0)} \qquad \mathcal{E}_M(k) \underset{k}{\bullet} A^{(k)}$$

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Smaller bar constructions

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Note on monoidality in manifold variable

$$\mathcal{E}_M(m) \times \mathcal{E}_N(n) \to \mathcal{E}_{M \amalg N}(m+n) \implies B(\mathcal{E}_M, \mathcal{D}, A) \wedge B(\mathcal{E}_N, \mathcal{D}, A) \to B(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

Monoidal but not symmetric

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Monoidal but not symmetric

Smaller bar constructions

Bar construction $\tilde{B}(\mathcal{E}_M, \mathcal{D}, A)$

$$\tilde{B}_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k) \ \underset{k}{\overset{\circlearrowleft}{\bullet}} \ A^{(k)}$$

$$\tilde{B}_1 = \bigvee_{k_0,k_1} (\mathcal{E}_M(k_1) \underset{\Sigma_{k_1}}{\times} \mathcal{D}(k_0,k_1))_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_1) \ \underset{k_1}{\overset{\circlearrowleft}{\bullet}} \leftarrow \underset{k_0}{\overset{\circlearrowleft}{\bullet}} \ A^{(k_0)}$$

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Smaller bar constructions

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$$\tilde{B}_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k) \ \underset{k}{\overset{\circlearrowright}{\bullet}} \ A^{(k)}$$

$$\tilde{B}_1 = \bigvee_{k_0, k_1} (\mathcal{E}_M(k_1) \underset{\Sigma_{k_1}}{\times} \mathcal{D}(k_0, k_1))_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_1) \ \underset{k_1}{\overset{\circlearrowright}{\bullet}} \leftarrow \underset{k_0}{\overset{\circlearrowright}{\bullet}} \ A^{(k_0)}$$

$$\tilde{B}_2 = \bigvee_{k_0, k_1, k_2} (\mathcal{E}_M(k_2) \underset{\Sigma_{k_2}}{\times} \mathcal{D}(k_1, k_2) \underset{\Sigma_{k_1}}{\times} \mathcal{D}(k_0, k_1))_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_2) \ \underset{k_2}{\overset{\circlearrowright}{\bullet}} \leftarrow \underset{k_1}{\overset{\circlearrowright}{\bullet}} \leftarrow \underset{k_0}{\overset{\circlearrowright}{\bullet}} \ A^{(k_0)}$$

etc

$$\tilde{B}(\mathcal{E}_M, \mathcal{D}, A) = B^\circ(\mathcal{E}_M, D, A) \qquad (D(m) = \mathcal{D}(m, 1))$$

Note on monoidality in manifold variable

$$\mathcal{E}_M(m) \times \mathcal{E}_N(n) \to \mathcal{E}_{M \amalg N}(m + n) \quad \Longrightarrow \quad B(\mathcal{E}_M, \mathcal{D}, A) \wedge B(\mathcal{E}_N, \mathcal{D}, A) \to B(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

Monoidal but not symmetric

Smaller bar constructions

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Monoidal but not symmetric

$$(\mathcal{E}_M(m) \times \mathcal{E}_N(n)) \underset{\Sigma_m \times \Sigma_n}{\times} \Sigma_{m+n} \xrightarrow{\cong} \mathcal{E}_{M\amalg N}(m+n) \implies \tilde{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \tilde{B}(\mathcal{E}_N, \mathcal{D}, A) \xrightarrow{\cong} \tilde{B}(\mathcal{E}_{M\amalg N}, \mathcal{D}, A)$$

Strong symmetric monoidal

Smaller bar constructions

Bar construction $\bar{B}(\mathcal{E}_M, \mathcal{D}, A)$

$$\bar{B}_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \underset{\Sigma_{k_0} \wr O(d)}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k) \overset{\circlearrowleft}{\underset{k}{\bullet}} A^{(k)}$$

$$\bar{B}_1 = \bigvee_{k_0,k_1} (\mathcal{E}_M(k_1) \underset{\Sigma_{k_1} \wr O(d)}{\times} \mathcal{D}(k_0, k_1))_+ \underset{\Sigma_{k_0} \wr O(d)}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_1) \overset{\circlearrowleft}{\underset{k_1}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_0}{\bullet}} A^{(k_0)}$$

$$\bar{B}_2 = \bigvee_{k_0,k_1,k_2} (\mathcal{E}_M(k_2) \underset{\Sigma_{k_2} \wr O(d)}{\times} \mathcal{D}(k_1, k_2) \underset{\Sigma_{k_1} \wr O(d)}{\times} \mathcal{D}(k_0, k_1))_+ \underset{\Sigma_{k_0} \wr O(d)}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_2) \overset{\circlearrowleft}{\underset{k_2}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_1}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_0}{\bullet}} A^{(k_0)}$$

etc

$$\tilde{B}(\mathcal{E}_M, \mathcal{D}, A) = B^{\circ}(\mathcal{E}_M, D, A) \qquad (D(m) = \mathcal{D}(m, 1))$$

$$\mathrm{Aut}_{\mathcal{D}}(k) = \Sigma_k \wr O(d)$$

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$$\mathcal{E}_M(m) \times \mathcal{E}_N(n) \to \mathcal{E}_{M \amalg N}(m+n) \implies B(\mathcal{E}_M, \mathcal{D}, A) \wedge B(\mathcal{E}_N, \mathcal{D}, A) \to B(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

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$$(\mathcal{E}_M(m) \times \mathcal{E}_N(n)) \underset{\Sigma_m \times \Sigma_n}{\times} \Sigma_{m+n} \xrightarrow{\cong} \mathcal{E}_{M \amalg N}(m+n) \implies \tilde{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \tilde{B}(\mathcal{E}_N, \mathcal{D}, A) \xrightarrow{\cong} \tilde{B}(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

Strong symmetric monoidal

The two-sided bar construction and partitions

Fix M, n and look at $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-))$

$$\bar{B}_0 = \coprod_{k_0} \mathcal{E}_M(k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0)$$

$$\bar{B}_1 = \coprod_{k_0,k_1} \mathcal{E}_M(k_1) \underset{\Sigma_{k_1} \wr O(d)}{\times} \mathcal{D}(k_0, k_1) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0)$$

$$\vdots \qquad \vdots$$

$$\bar{B}_r = \coprod_{k_0,\ldots,k_r} \mathcal{E}_M(k_r) \underset{\Sigma_{k_r} \wr O(d)}{\times} \mathcal{D}(k_{r-1}, k_r) \underset{\Sigma_{k_{r-1}} \wr O(d)}{\times} \cdots \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0)$$

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For $P_\bullet = (P_r \leq \cdots \leq P_0)$, define subspace $\mathcal{E}(P_\bullet)$. Can recover $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-))$ from these.

$$\mathcal{E}_M(k_r) \underset{\Sigma_{k_r} \wr O(d)}{\times} \mathcal{D}(k_{r-1}, k_r) \underset{\Sigma_{k_{r-1}} \wr O(d)}{\times} \cdots \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0) = \coprod_{\substack{P_r \leq \cdots \leq P_0 \\ \# P_i = k_i}} \mathcal{E}(P_r \leq \cdots \leq P_0)$$

Center point map $\mathcal{E}(P_\bullet) \to \mathcal{E}_M(n) \to C_{Gl(d)}(n, M)$

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Large scale open sets associated to partitions

Idea: find open subsets $U(P_\bullet)$ of $C_{Gl(d)}(n, M)$
that "look like" $\mathcal{E}(P_\bullet)$ via center point map

$$\mathcal{E}(P_\bullet) \overset{\simeq}{\leftarrow} E(P_\bullet) \overset{\simeq}{\rightarrow} U(P_\bullet) \subset C_{Gl(d)}(n, M)$$

Choose a nice Riemannian metric on M
(Convexity radius ≥ 1, sectional curvatures between -1 and 1)

Choose a large parameter λ to use for scale

Idea for $U(P_\bullet)$

Notation: $(\vec{x}_1, \ldots, \vec{x}_n) \in C_{Gl(d)}(n, M)$, image $(x_1, \ldots, x_n)$ in $C(n, M)$

- If i, j are in same block of P_ℓ, then $d(x_i, x_j) < \lambda^{-2k_\ell}$
- If i, j are in different blocks of P_ℓ, then $d(x_i, x_j) > \lambda^{-2k_\ell + 1}$

Idea for $E(P_\bullet)$

- Restrict to nice embeddings in $\mathcal{E}_M(k_r)$, close to exp on a disk of fixed radius $(r = \frac{2}{5}\lambda^{-2k_r})$
- Require center points of the embeddings to land minimum distance apart $(\frac{4}{3}\lambda^{-2k_r+1})$
- Require the disks in $\mathcal{D}(k_\ell, k_{\ell+1})$ to have fixed radius $(r = \lambda^{-2(k_\ell - k_{\ell+1})})$
and center points to be a minimum distance apart $(4\lambda^{-2(k_\ell - k_{\ell+1}+1)})$

Why k's instead of ℓ? Compatible with insertion/deletion

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Idea for $E(P_\bullet)$

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The two-sided bar construction and the Čech complex

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Observations

- Every element of $C_{Gl(d)}(n, M)$ is in some $U(P)$
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In other words:

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On geometric realization we get

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n, -)) = |\coprod \mathcal{E}(P_\bullet)| \xleftarrow{\simeq} |\coprod E(P_\bullet)| \xrightarrow{\simeq} |\coprod U(P_\bullet)|$$

This “proves” center point map $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n, -)) \to \mathcal{E}_M(n)$ is an equivalence

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Quillen's Theorem A Criterion

Setting

$$\begin{array}{cccc} \mathcal{E}_M & \mathcal{D} & A & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow & \downarrow f & \| & \downarrow \\ \mathcal{Y} & \mathcal{X} & A & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

Observations

- $\bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) \cong \bar{B}(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A)$
- For fixed $x \in X$, $\bar{B}(\mathcal{X}(-, x), \mathcal{X}, A) \to A(x)$ is an equivalence
- $\bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) \to \bar{B}(\mathcal{E}_M, \mathcal{D}, A)$ is an equivalence

Map $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(x, f(-))) \to \bar{B}(\mathcal{Y}, \mathcal{X}, \mathcal{X}(x, -)) \to \mathcal{Y}(x)$ induces map on homotopy coend

$$\begin{array}{ccc} \bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) & \xrightarrow{\simeq} & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \cong\downarrow & & \vdots \\ \bar{B}(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A) & \longrightarrow & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Theorem (Quillen's Theorem A)

If $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(x, f(-))) \to \mathcal{Y}(x)$ is an equivalence for every $x \in \mathcal{X}$, then $\bar{B}(\mathcal{E}_M, \mathcal{D}, A) \to \bar{B}(\mathcal{Y}, \mathcal{X}, A)$ is an equivalence.

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Theorem (Quillen's Theorem A)

If $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(x, f(-))) \to \mathcal{Y}(x)$ is an equivalence for every $x \in \mathcal{X}$, then $\bar{B}(\mathcal{E}_M, \mathcal{D}, A) \to \bar{B}(\mathcal{Y}, \mathcal{X}, A)$ is an equivalence.

Quillen's Theorem A Criterion

Setting

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$$\begin{array}{ccc} \bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) & \xrightarrow{\simeq} & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \cong \downarrow & & \vdots \downarrow \\ \bar{B}(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A) & \longrightarrow & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Theorem (Quillen's Theorem A)

If $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(x, f(-))) \to \mathcal{Y}(x)$ is an equivalence for every $x \in \mathcal{X}$, then $\bar{B}(\mathcal{E}_M, \mathcal{D}, A) \to \bar{B}(\mathcal{Y}, \mathcal{X}, A)$ is an equivalence.

Quillen's Theorem A Criterion

Setting

$$\begin{array}{ccccc} \mathcal{E}_M & \mathcal{D} & A & & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow & \downarrow f & \| & & \downarrow \\ \mathcal{Y} & \mathcal{X} & A & & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

Observations

- $\bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) \cong \bar{B}(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A)$
- For fixed $x \in X$, $\bar{B}(\mathcal{X}(-, x), \mathcal{X}, A) \to A(x)$ is an equivalence
- $\bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) \to \bar{B}(\mathcal{E}_M, \mathcal{D}, A)$ is an equivalence

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If $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(x, f(-))) \to \mathcal{Y}(x)$ is an equivalence for every $x \in \mathcal{X}$, then $\bar{B}(\mathcal{E}_M, \mathcal{D}, A) \to \bar{B}(\mathcal{Y}, \mathcal{X}, A)$ is an equivalence.

Quillen's Theorem A Criterion

Setting

$$\begin{array}{ccccc} \mathcal{E}_M & \mathcal{D} & A & & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow & \downarrow f & \| & & \downarrow \\ \mathcal{Y} & \mathcal{X} & A & & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

Observations

- $\bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) \cong \bar{B}(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A)$
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Quillen's Theorem A Criterion

Setting

$$\begin{array}{ccccc} \mathcal{E}_M & \mathcal{D} & A & & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow & \downarrow f & \| & & \downarrow \\ \mathcal{Y} & \mathcal{X} & A & & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

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Observations

- $\bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) \cong \bar{B}(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A)$
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Quillen's Theorem A Criterion

Setting

$$\begin{array}{cccc} \mathcal{E}_M & \mathcal{D} & A & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow & \downarrow f & \| & \downarrow \\ \mathcal{Y} & \mathcal{X} & A & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

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- $\bar{B}(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)) \cong \bar{B}(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A)$
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Quillen's Theorem A Criterion

Setting

$$\begin{array}{ccc} \mathcal{E}_M & \mathcal{D} & A \\ \downarrow & \downarrow f & \| \\ \mathcal{Y} & \mathcal{X} & A \end{array} \qquad \begin{array}{c} \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow \\ \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

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Quillen's Theorem A Criterion

Setting

$$\begin{array}{ccccc} \mathcal{E}_M & \mathcal{D} & A & & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow & \downarrow{\scriptstyle f} & \| & & \downarrow \\ \mathcal{Y} & \mathcal{X} & A & & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

Observations

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Quillen's Theorem A Criterion

Setting

$$\begin{array}{ccc} \mathcal{E}_M & \mathcal{D} & A \\ \downarrow & \downarrow f & \| \\ \mathcal{Y} & \mathcal{X} & A \end{array} \qquad \begin{array}{c} \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow \\ \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\quad \mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

Observations

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$$\begin{array}{ccc} \bar{B}\big(\mathcal{E}_M, \mathcal{D}, \bar{B}(\mathcal{X}(-, f(-)), \mathcal{X}, A)\big) & \xrightarrow{\simeq} & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \cong \downarrow & & \vdots \downarrow \\ \bar{B}\big(\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{X}(-, f(-))), \mathcal{X}, A\big) & \longrightarrow & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

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Quillen's Theorem A Criterion

Setting

$$\begin{array}{ccccccc} \mathcal{E}_M & & \mathcal{D} & & A & & \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \\ \downarrow & & \downarrow f & & \| & & \downarrow \\ \mathcal{Y} & & \mathcal{X} & & A & & \bar{B}(\mathcal{Y}, \mathcal{X}, A) \end{array}$$

Double homotopy coend trick $\mathcal{E}_M \circlearrowright \mathcal{D} \circlearrowleft \mathcal{X}(-, f(-)) \circlearrowright \mathcal{X} \circlearrowleft A$

Observations

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Examples

Unital vs non-unital

If A is unital, have homotopy left Kan extension for unital categories, and comparison map

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A) \to \bar{B}(\mathcal{E}^u_M, \mathcal{D}^u, A)$$

Need to check $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^u(n, -)) \xrightarrow{\simeq} \mathcal{E}^u_M(n)$

Symmetric monoidality in algebra variable

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A \wedge B) \to \bar{B}(\mathcal{E}_M \times \mathcal{E}_M, \mathcal{D} \times \mathcal{D}, A \wedge B) \cong \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \bar{B}(\mathcal{E}_M, \mathcal{D}, B)$$

Comparison map $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(m, -) \times \mathcal{D}(n, -)) \to \mathcal{E}_M(m) \times \mathcal{E}_M(n)$ is **not** a weak equivalence

Map itself is not a weak equivalence in general, except in unital case:

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A \wedge B) \to \bar{B}(\mathcal{E}^u_M \times \mathcal{E}^u_M, \mathcal{D}^u \times \mathcal{D}^u, A \wedge B) \cong \bar{B}(\mathcal{E}^u_M, \mathcal{D}^u, A) \wedge \bar{B}(\mathcal{E}^u_M, \mathcal{D}^u, B)$$

Comparison map $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^u(m, -) \times \mathcal{D}^u(n, -)) \xrightarrow{\simeq} \mathcal{E}^u_M(m) \times \mathcal{E}^u_M(n)$

... ...

Let's look at similar but easier comparison

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m, n), -)) \xrightarrow{\simeq} \mathcal{E}^2_M(m, n)$$

where $\mathcal{D}^2((m, n), k) \subset \mathcal{D}^u(m, k) \times \mathcal{D}^u(n, k)$, $\mathcal{E}^2_M(m, n) \subset \mathcal{E}^u_M(m) \times \mathcal{E}^u_M(n)$ (surjective on π_0 maps)

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Symmetric monoidality in algebra variable

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Symmetric monoidality in algebra variable

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Comparison map $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^u(m, -) \times \mathcal{D}^u(n, -)) \xrightarrow{\simeq} \mathcal{E}_M^u(m) \times \mathcal{E}_M^u(n)$

Let's look at similar but easier comparison

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m, n), -)) \xrightarrow{\simeq} \mathcal{E}_M^2(m, n)$$

where $\mathcal{D}^2((m, n), k) \subset \mathcal{D}^u(m, k) \times \mathcal{D}^u(n, k)$, $\mathcal{E}_M^2(m, n) \subset \mathcal{E}_M^u(m) \times \mathcal{E}_M^u(n)$ (surjective on π_0 maps)

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Symmetric monoidality in algebra variable

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. . .   . . .

Let's look at similar but easier comparison

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m, n), -)) \xrightarrow{\simeq} \mathcal{E}^2_M(m, n)$$

where $\mathcal{D}^2((m, n), k) \subset \mathcal{D}^u(m, k) \times \mathcal{D}^u(n, k)$, $\mathcal{E}^2_M(m, n) \subset \mathcal{E}^u_M(m) \times \mathcal{E}^u_M(n)$ (surjective on π_0 maps)

Examples

Unital vs non-unital

If A is unital, have homotopy left Kan extension for unital categories, and comparison map

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A) \to \bar{B}(\mathcal{E}_M^u, \mathcal{D}^u, A)$$

Need to check $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^u(n, -)) \xrightarrow{\simeq} \mathcal{E}_M^u(n)$

Symmetric monoidality in algebra variable

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A \wedge B) \to \bar{B}(\mathcal{E}_M \times \mathcal{E}_M, \mathcal{D} \times \mathcal{D}, A \wedge B) \cong \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \bar{B}(\mathcal{E}_M, \mathcal{D}, B)$$

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Example: $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m,n),-)) \xrightarrow{\simeq} \mathcal{E}^2_M(m,n)$

$$\bar{B}_0 = \coprod_{k_0} \mathcal{E}_M(k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}^2((m,n),k_0)$$

$$\bar{B}_1 = \coprod_{k_0,k_1} \mathcal{E}_M(k_1) \underset{\Sigma_{k_1} \wr O(d)}{\times} \mathcal{D}(k_1,k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}((m,n),k_0)$$

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$$\bar{B}_r = \coprod_{k_0,\dots,k_r} \mathcal{E}_M(k_r) \underset{\Sigma_{k_r} \wr O(d)}{\times} \mathcal{D}(k_{r-1},k_r) \underset{\Sigma_{k_{r-1}} \wr O(d)}{\times} \cdots \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}((m,n),k_0)$$

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Now partitions of $m+n$

Define $U(P_\bullet) \subset C_{Gl(d)}(m,M) \times C_{Gl(d)}(n,M)$ by exactly the same distance conditions

Define $E(P_\bullet)$ by exactly the same radius and center point conditions
(But now outer little disks are in $\mathcal{D}^2((m,n),k_0)$, so the first m disks can overlap with the last n)

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