Trace Methods in Algebraic K-theory

Reduce relative K-theory calculations to (equivalent) stable homotopy theory.

Given algebraic K-theory input → K-theory calculations

Examples (non-comprehensive)

Hesselholt–Madsen 97:

Let $k$ be a perfect field

$K(W_k)^p = TC(W_k)^p \otimes \kappa(0)$

$k(k)^p = H\mathbb{Z}_p$.

$K(W_k)^p \cong TC(W_k)^p \otimes \kappa(0)$ is finite algebra over $W_k$.

Söker–Madsen 02:

$\mathbb{Z}_p^2 \cong W\mathbb{F}_p$

$k(Z_p)^p = \langle j, v \mid j^2 = j, v^2 = k \rangle_p$
Hesselholt-Madsen '03

\[ K(F), \ K(\mathcal{O}_K) \quad F \text{ is a finite extension of } \mathbb{Q}_p \]

Rognes '02, '03, Blumberg-Mandell '20

Input (Rognes-Weibel) calculation of \( K(\mathbb{Z}) \)

(plus \( A(1) \), Voevodsky, Rost)

\( K(\mathbb{S})[1/2] \)

\( \text{after inverting 2} \)

\[ 0 \to K_*(\mathbb{S}) \to K_*(\mathbb{S}) \otimes K_*(\mathbb{G}_m) \otimes K_{\mathcal{O}}(\mathbb{Z}) \to K_{\mathcal{O}}(\mathbb{Z}_p) \to 0 \]

is exact \( \# \mathcal{O} \) in fact split exact.

All this stems from fundamental work on

Goodwillie '86, McCarthy '93, Dundas '97

about natural trans \( K \to TC \).
Let \( R \to S \) be a map of connective \( R \)-ring spectra (for example, rings or simplicial rings). Assume \( R \to S \) is surjective with no kernel.

Then

\[
\begin{align*}
K(R) & \to TC(R) \\
\downarrow & \downarrow \\
K(S) & \to TC(S)
\end{align*}
\]

is homotopy cartesian.

In other words

\[
\begin{align*}
K(R \to S) & := \text{Fib}(K(R) \to K(S)) \\
TC(R \to S) & := \text{Fib}(TC(R) \to TC(S))
\end{align*}
\]

is a weak equivalence. 

Goodwillie '86 native case for simplicial rings

McCarthy '97 \( \mathbb{p} \)-completion case for simplicial rings
TC is a theory built from THH

THH is topological Hochschild homology

\[ \text{THH}(A) \quad \left( \begin{array}{c}
A \xleftarrow{\partial} A \otimes A \xleftarrow{\partial} A \otimes A \otimes A \\
\text{and so on}
\end{array} \right) \]

\text{simplifies abelian group}

\[ \text{THH} \quad \left( \begin{array}{c}
A \xleftarrow{\partial} A \otimes A \otimes A \xleftarrow{\partial} A \otimes A \otimes A \otimes A \\
\text{cyclic}
\end{array} \right) \]

\text{simplifies spectrum}

\text{cyclic spectrum}

\text{cyclic (cones)} \rightarrow \text{get a } \mathbb{P}^1 = S^1 \text{ unit cx numbers}

Picture THH as! Look at coneg or n pts on S^1

\[ L_n A^{(n)} \otimes S^n \]

\[ \emptyset \text{ or } A^{(n)} \]

\[ \varphi : A^{(n)} \rightarrow A \]

Labelled by \( A^{(n)} = A_n \)

Identity by \( A^{(i)} \wedge \wedge A^{(j)} \)

\( (x_0, \ldots, x_k) \)

\( \sim A^{(i)} \wedge A^{(j)} \)

as points come together, multiply factors
$\phi^C_\eta : \text{THH}(\mathbf{A}) \to \text{THH}(\mathbf{A})$

$R : \text{THH}_{\mathbf{C}_0} \to \text{THH}_{\mathbf{C}_0^{\eta R}}$

$\text{TC} = \text{holim} \text{THH}_{\mathbf{C}_0^{\eta R}}$

$\text{ker} \phi \cong R$

$K \to \text{THH}$

$\uparrow$

$\rightarrow \text{TC}$. 
References

Survey Paper/Book


Foundational Papers for the Goodwillie-McCarthy-Dundas Theorem


Papers for K-Theory “Input”


Trace Method Calculations Papers Mentioned in the Talk