Cochains and Homotopy Theory

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Abstract

It is known that the E-infinity algebra structure on the cochain complex of a space contains all the homotopy theoretic information about the space, but for partial information, less structure is needed. I will discuss some ideas and preliminary work in this direction.

Outline

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1. Distinguishing homotopy types
2. Homotopy algebras and operadic algebras
3. Formality in characteristic p
4. Generalizing AHAH   Anick HAH.

Distinguishing Homotopy Types

Explanation through examples

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Example (1st Semester Algebraic Topology)

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Cannot distinguish $\mathbb{C}P^2$ and $S^2 \vee S^4$ just through homology, but can distinguish them through their cohomology with **cup product**.

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Example (3rd or 4th Semester Algebraic Topology)

Can classify homotopy types of all simply connected spaces with homology like $\mathbb{C}P^2$ or $S^2 \vee S^4$ their cohomology with cup product.

by

Example (4th Semester Algebraic Topology)

Cannot distinguish $\Sigma\mathbb{C}P^2$ and $S^3 \vee S^5$ through their cohomology just with cup product, but can distinguish them using the Sq^2

Steenrod operation

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Example (Advanced Graduate Algebraic Topology)

Homotopy types of spaces with homology like $S^n \vee S^{n+r}$ can be distinguished and classified using **higher cohomology operations** "named" by the E^2-term of the unstable Adams spectral sequence.

Cohomology and Homotopy Types

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- McClure–Smith, “Multivariable Cochain Operations and Little *n*-Cubes”, 2003

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Simply connected homotopy types are distinguished by the E_∞ structure on the cochains.

- Mandell, “Cochains and Homotopy Type”, 2006

Older Work

Rational Homotopy Theory

p-adic Homotopy Theory

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- Serre, 1950's

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B-C-K-Q-R-S

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- Adams, Mass. Inst. of Topology, 1958-1972
- Kriz, 1993
- Goerss, 1995
- Mandell, 2001

simplicial

cochain

Practical Remarks

Rational Homotopy Theory

p-Adic Homotopy Theory

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- Objects: Rational commutative differential graded algebras

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- Standard form: Cofibrant model
- Cofibrant model still very big, not always easy to work with
- For spaces close to $K(\pi, n)$'s, practical to find cofibrant model
- No notion of “formal” space

Further Directions

Explanatory examples show that you do not need to keep track of the whole E_∞ structure to say interesting things about classification of homotopy types.

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Approach: Weaken the algebraic structure

Look at an algebraic structure weaker than E_∞ and see what information is left.

Algebraic Structures

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Example: Limited Steenrod Operations

McClure–Smith show that sub-operad $\mathcal{S}_n$ coming from first (i.e., last) bunch of Steenrod operations is an E_n operad.

– Boardman Vogt / F. Cohen

– Berger-Fresse   Bar Operad   Barratt Eccles operad

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What information is left when we view C^*X as an E_n algebra?

What is left in the E_n Structure?

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  Can compute E_{n-1}-structure on $C^*(Y \overset{h}{\times}_X Z)$ as $\mathrm{Tor}^{C^*X}(C^*Y, C^*Z)$.

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An approach to formality in characteristic p.

Formality in Characteristic Zero

Definition

A commutative differential graded $\mathbb{Q}$-algebra is *formal* if it is quasi-isomorphic to its cohomology though maps of commutative differential graded algebras.

Examples

- A $\mathbb{Q}$-CDGA with zero differential is formal
- A $\mathbb{Q}$-CDGA whose cohomology is a free gr. com. algebra is formal
- A $\mathbb{Q}$-CDGA whose cohomology is an exterior algebra is formal

Definition

A space is *rationally formal* if its polynomial De Rham complex is a formal $\mathbb{Q}$-CDGA.

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Examples of Rationally Formal Spaces

Spheres

Cohomology is an exterior algebra.

Lie Groups / H-Spaces

Milnor-Moore: Cohomology is a free gr. comm. algebra.

Wedges and Products of Formal Spaces

Complex Algebraic Varieties

Deligne-Griffiths-Morgan-Sullivan / Morgan

Mixed Hodge structure on cohomology gives a mixed Hodge structure on the De Rham complex. Limits possibilities for differentials.

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Formality for E_∞ Algebras

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Cohomology of an E_∞ algebra has E_∞ algebra from its graded commutative algebra structure.

In characteristic p, cohomology of E_∞ algebras have Steenrod / Dyer-Lashof operations. For commutative algebras, all but the p-th power operation are zero.

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For spaces, the zeroth operation is the identity.

The cochain algebra of a space cannot be formal unless the space has contractible components.

E_n Algebras

E_n algebras have operations on $x \in H^*A$

$$Sq^m x, Sq^{m-1} x, \ldots, Sq^{m-n+1} x \qquad p = 2, |x| = m$$

$$P^m x, P^{m-1} x, \ldots, P^{m-\lfloor (n-1)/2 \rfloor} x \qquad p > 2, |x| = 2m$$

$$P^m x, P^{m-1} x, \ldots, P^{m-\lfloor n/2 \rfloor} x \qquad p > 2, |x| = 2m+1, n > 1$$

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If X is an $(n-1)$-connected space, no Sq^0/P^0 operation in E_n structure on cochains

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Which $(n-1)$-connected spaces are E_n formal?

Loops and Suspension

Recall: For any space X, Ω^n is an E_n space.

If X is an E_{n-1}-space, ΩX is an E_n space.

Because C^* is contravariant, $C^*\Sigma X$ is "like" loops of C^*X.
(Think $H\mathbb{Z}^{\Sigma X} \cong \Omega H\mathbb{Z}^X$)

Theorem

The E_{n-1} structure on C^*X determines the E_n structure on $C^*\Sigma X$.

Consequence

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Conjecture (Formality)

If X is rationally formal, then after inverting finitely many primes, C^*X is E_n formal.

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Theorem (Anick)

Let R be a ring containing $1/m$ for $m < p$ and let X be an r-connected pr-dimensional CW complex. Then the Adams-Hilton model of X with coefficients in R is the universal enveloping algebra of a Lie algebra.

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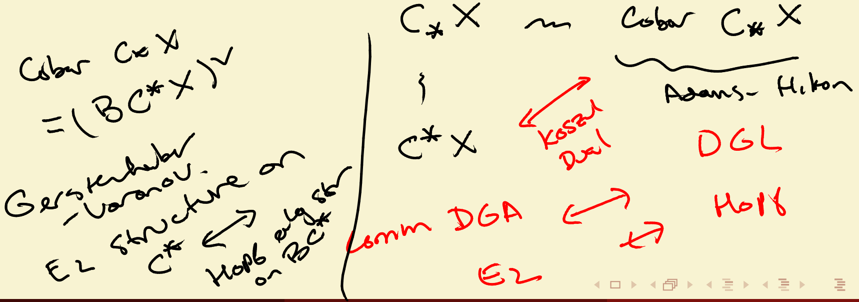

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Let R be a ring containing $1/m$ *for* $m < p$ *and let* X *be an* r*-connected* pr*-dimensional simplicial complex. Then* $C^*(X; R)$ *is quasi-isomorphic as an* E_2*-algebra to a commutative differential graded algebra.*

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For a 2-connected space, after inverting finitely many primes, $C^*(X)$ is quasi-isomorphic as an E_2-algebra to a commutative differential graded algebra.

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Can do homotopy theory with $\mathcal{C}om'$ algebras.

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land in degrees $k(r+1) - (k-1) = kr + 1$ and above.

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We can look at obstruction theory for the $\mathcal{S}_2'$-structure to extend to a $\mathcal{C}om'$-structure.

The Linearity Hypothesis

Hypothesis. There exists and E_n operad $\mathcal{E}$ that acts on cochain complexes and satisfies the dimension bound

$$\dim \mathcal{E}(k) = (k-1)(n-1).$$

Highest chain-level k-ary operation occurs in degree $(k-1)(n-1)$.

Notes.

- This is the same degree as highest non-zero homology group.
- The standard E_n operads satisfy this bound for $k = 2$.
- Standard E_1 and E_2 operads satisfy this bound for all n, k.
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A Weaker Linearity Hypothesis

At the cost of weakening the conjectures, the hypothesis can be weakened to a linearity hypothesis

$$\dim \mathcal{E}(k) = a(k-1)(n-1) \qquad \text{for } k \gg 0$$

The little n-cubes operad of spaces has k-th space a non-compact manifold with boundary, dimension $k(n+1)$.

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S_3 grows like linear: inverse of ackermans

Consequences of the Linearity Hypothesis

Let X be r-reduced dimension d, so $\tilde{C}^* X = 0$ for $* \leq r$ and $* > d$

Look at E_n action.

$$\mathcal{E}(k) \otimes (\tilde{C}^* X)^{\otimes k} \to \tilde{C}^* X.$$

Left side is non-zero in range $k(r+1) - (k-1)(n-1)$ to kd.
Right side is non-zero in range $r+1$ to d.

$k(r+1)-(k-1)(n-1) = k(r+1-(n-1))+(n-1) = k(r-n+2)+n-1$

So if $k(r-n+2)+n-1 > d$ the map must be zero.

Limit dimension to $p(r-n+2)-n-2$ or even $p(r-n+2)$.

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Generalizing Anick's HAH Theorem

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