The Homotopy Theory of Cyclotomic Spectra

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Midwest Topology Seminar

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Overview

The homotopy theory of cyclotomic spectra

- Joint work with Andrew Blumberg

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   THH and *TC*

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   THH and *TC*
2. What is a cyclotomic spectrum?

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   THH and *TC*
2. What is a cyclotomic spectrum?
   Some equivariant stable homotopy theory

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1. How do cyclotomic spectra show up?
   THH and *TC*
2. What is a cyclotomic spectrum?
   Some equivariant stable homotopy theory
3. The homotopy theory of cyclotomic spectra
4. A new interpretation of *TC*

Hochschild Homology

Cyclic bar construction

$$N^{cy}_q R = \underbrace{R \otimes \cdots \otimes R}_{q \text{ factors}} \otimes R$$

$$\begin{array}{ccc} & R \otimes \cdots \otimes R & \\ \otimes & & \otimes \\ & R & \end{array}$$

Example: $R = \mathbb{Z}[\pi]$ $\quad \pi = \pi_1 X.$

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$$\otimes \qquad\qquad\qquad \otimes$$

$$R$$

Example: $R = \mathbb{Z}[\pi]$

Chain complex

Hochschild Homology HH_*

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$\otimes$ $\otimes$

R

Example: $R = \mathbb{Z}[\pi]$

Chain complex

Cyclic structure $\Longrightarrow$ Connes' B operator

Example
$HH_*(\mathbb{Z}[\pi])$
$= H_*(\Lambda B\pi)$

$\pi = \pi_1 X$ X negatively curved manifold

$HH_*(\mathbb{Z}[\pi]) = H_*(\Lambda X)$.

Cyclic homology HC
Periodic cyclic homology HP
negative cyclic homology
$HC^- = HN$

Topological Hochschild Homology

Cyclic bar construction

$THH(\Sigma^\infty \Omega_+)$
$= \Sigma^\infty (\Lambda X)_+$

$$N^{cy}_q R = \underbrace{R \wedge \cdots \wedge R}_{q \text{ factors}} \wedge R$$

$$R \wedge \cdots \wedge R$$

$$\wedge \qquad\qquad \wedge$$

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$\pi = \pi_1 X$

Ω group like π

$\Omega = \Omega X.$

Example: $R = H\mathbb{Z}[\pi]$ or $R = \Sigma^\infty \Omega_+$

Spectrum

Cyclic structure $\Longrightarrow$

Topological Hochschild Homology

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Example: $R = H\mathbb{Z}[\pi]$ or $R = \Sigma^\infty \Omega_+$

Spectrum
Cyclic structure $\Longrightarrow$ circle group action

The Dennis Trace and Goodwillie's Theorem

Trace map $K_*(R) \to HH_*(R)$.

Factors through negative cyclic homology

$$K_*(R) \to HN_*(R)$$

For a map $A \to B$, get a map on relative theories

$$K_*(B, A) \to HN_*(B, A)$$

Theorem (Goodwillie, 1986)

Let $A \to \bar{B}$ be a surjection with nilpotent kernel. Then the map on relative theories

$$K_*(\bar{B}, A) \to HN_*(\bar{B}, A)$$

is an isomorphism after tensoring with $\mathbb{Q}$

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The Cyclotomic Trace and McCarthy/Dundas Theorems

Question (Goodwillie)

Is there a corresponding theory that gets the p-adic information?

Bökstedt–Hsiang–Madsen constructed spectrum TC
and map

$$K(R) \to TC(R)$$

Conjecture (1989)

Let $A \to B$ be a map of rings
and assume it is surjective with nilpotent kernel
then $K(B, A) \to TC(B, A)$ is a weak equivalence after p-completion.

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$$K(R) \to TC(R)$$ Cyclotomic trace

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Theorem (McCarthy/Dundas, 1997)

Let $A \to B$ be a map of ring spectra connective
and assume it is surjective with nilpotent kernel on π_0
then $K(B, A) \to TC(B, A)$ is a weak equivalence after p-completion.

$K(\mathbb{F}_p)^\wedge_p = H\mathbb{Z}^\wedge_p$ $\quad K(\mathbb{Z}^\wedge_p)^\wedge_p = H\mathbb{Z}^\wedge_p$

TC is not THN

	Algebra	Topology
Abelian Groups	HH_*	THH_*
Spectrum	HH	THH
	HN_*	
	HN	

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TC is not THN

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Homotopy Fixed Points	HN_*	$THH^*_{\mathbb{T}}(E\mathbb{T})$
Homotopy Fixed Point Spectrum	HN	$THH^{h\mathbb{T}}$

$$TC \neq THH^{h\mathbb{T}}$$

Instead involves **fixed points** and **geometric fixed points** for finite subgroups C_n

$\mathbb{T}$ = circle group

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Instead involves **fixed points** and **geometric fixed points** for finite subgroups C_{p^m}

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Fixed Points and Geometric Fixed Points

Example

For a $\mathbb{T}$-space X, what are the C_p-fixed points of $\Sigma^\infty X_+$?

Might want/expect them to be $\Sigma^\infty X_+^{C_p}$

Geometric Fixed Points

The **geometric fixed point functor** Φ^{C_p} has the property that

$$\Phi^{C_p}\Sigma^\infty X_+ = \Sigma^\infty \quad X_+^{C_p}$$

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Fixed Points

Look at the case of $S = \Sigma^{\infty}_{\mathbb{T}} S^0$.
S^{C_p} = spectrum of C_p-equivariant maps from S to S.

$\Phi^{C_p} S = \Sigma^{\infty}_{\mathbb{T}/C_p} S^{C_p} = \underset{\mathbb{T}/C_p}{\underset{\sim}{S}}$

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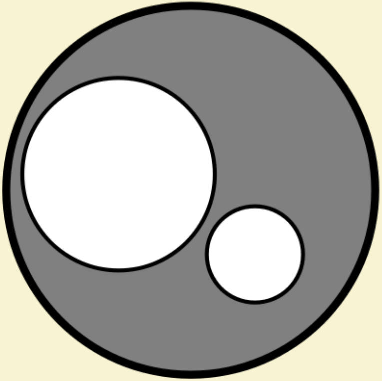
Non-equivariant map example



$S^2 \to S^2$

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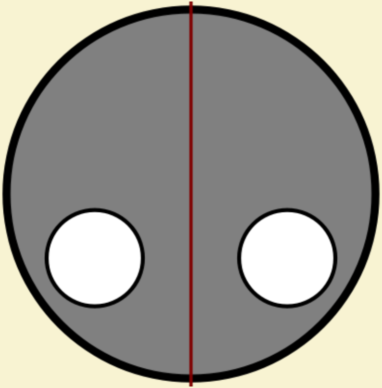
C_2-equivariant map example



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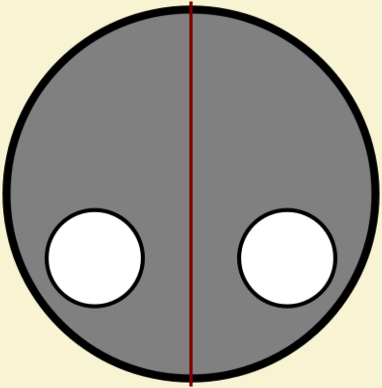
C_2-equivariant map examples



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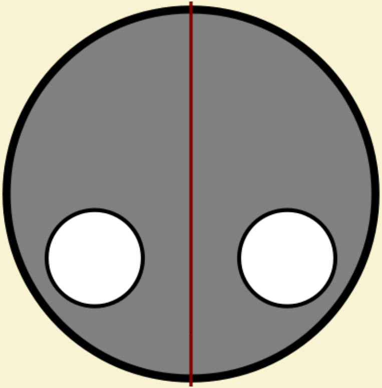
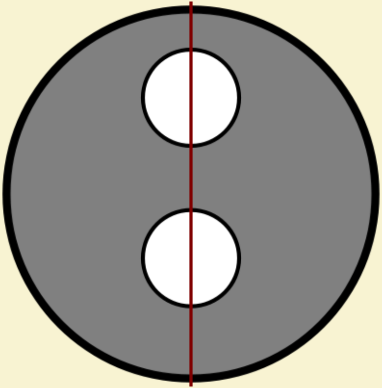
$\pi_0(S^{C_p})$ G-sets, formal add inverses

Burnside ring

C_2-equivariant map examples





Fixed Points and Geometric Fixed Points

Canonical map

$$T^{C_p} \to \Phi^{C_p} T$$

For suspension spectra, this map is split (tom Dieck splitting).
Other piece is suspension spectrum of C_p homotopy orbits.

Summary

For $T = \Sigma^\infty_{\mathbb{T}} X_+$

- Geometric fixed points $\Phi^{C_p} T = \Sigma^\infty_{\mathbb{T}/C^p} X^{C^p}_+$
- Fixed points $T^{C_p} = \Sigma^\infty_{\mathbb{T}/C^p} (X^{C^p} \amalg (E\mathbb{T} \times_{C_p} X))_+$

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$$T^{C_{p^2}} = \Sigma^\infty_{\mathbb{T}/C^{p^2}} (X^{C^p} \amalg (E(\mathbb{T}/C_p) \times_{C_{p^2}/C_p} X^{C_p}) \amalg (E\mathbb{T} \times_{C_{p^2}} X))_+$$

The Structure of *THH*

Example

$R = \Sigma^\infty \Omega_+$, $\Omega = \Omega X$.

$$THH(R) \simeq \Sigma^\infty_{\mathbb{T}} \Lambda X$$

$\Phi^{C_p} THH \simeq \Sigma^\infty_{\mathbb{T}/C_p} (\Lambda X)^{C_p}$

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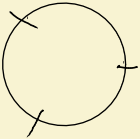

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$$\rho^*(\Lambda X)^{C_p} \cong \Lambda X \quad \Longrightarrow \quad \rho^* \Sigma^\infty_{\mathbb{T}/C_p} (\Lambda X)^{C_p} \simeq \Sigma^\infty_{\mathbb{T}} \Lambda X$$

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Cyclotomic Structure

$$r \colon \rho^* \Phi^{C_p} THH(R) \xrightarrow{\simeq} THH(R)$$

$\rho \colon \mathbb{T} \cong \mathbb{T}/C_p$

Construction of TC

The maps R and F

$$R\colon THH^{C_{p^m}} \to (\rho^* \Phi^{C_p} THH)^{C_{p^{m-1}}} \to THH^{C_{p^{m-1}}}$$

$$F\colon THH^{C_{p^m}} \to THH^{C_{p^{m-1}}}$$

Definition

$$TR = \operatorname{holim}(\cdots \xrightarrow{R} THH^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} THH)$$

$$TF = \operatorname{holim}(\cdots \xrightarrow{F} THH^{C_{p^m}} \xrightarrow{F} \cdots \xrightarrow{F} THH)$$

$$TC = TR^{hF} \simeq TF^{hR}$$

Construction of *TC*

The maps *R* and *F*

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Construction of *TC*

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$$TF = \text{holim}(\cdots \xrightarrow{F} THH^{C_{p^m}} \xrightarrow{F} \cdots \xrightarrow{F} THH)$$

$$TC = TR^{hF} \simeq TF^{hR}$$

Construction of *TC*

The maps *R* and *F*

$$R\colon THH^{C_{p^m}} \to (\rho^* \Phi^{C_p} THH)^{C_{p^{m-1}}} \to THH^{C_{p^{m-1}}}$$

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The Homotopy Theory of Cyclotomic Spectra

The Homotopy Theory of Cyclotomic Spectra

1. Cofiber sequences

   - **Hesselholt-Madsen**: Computation of K-theory of local fields and proof of the Quillen-Lichtenbaum conjecture
   - **Ausoni-Rognes**: Program for understanding $A(*)$
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2. What are maps of cyclotomic spectra?
   - Strictly commuting structure maps?
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Homotopy Theory

Model Category

Consists of a category $\mathcal{C}$ having all finite limits and colimits
Together with three classes of maps, called **cofibrations**, **fibrations**, and **weak equivalences**
Such that:

1. Weak equivalences satisfy the 2-out-of-3 property
2. All three classes of maps are closed under retracts
3. Cofibrations have the left lifting property with respect to acyclic fibrations and fibrations have the left lifting property with respect to acyclic cofibrations
4. Every map factors as a cofibration followed by an acyclic fibration and as an acyclic cofibration followed by a fibration

$\Longrightarrow$ abstract homotopy theory / homotopy category / good theory of derived functors / etc., etc., etc.

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The Model* Category of Cyclotomic Spectra

Definition

A pre-cyclotomic spectrum is an orthogonal $\mathbb{T}$-spectrum T together with a map of orthogonal $\mathbb{T}$-spectra

$$\rho^* \Phi^{C_p} T \to T.$$

A cyclotomic spectrum is a pre-cyclotomic spectrum for which the structure map is* a weak equivalence.

A map of pre-cyclotomic spectra is a map of orthogonal $\mathbb{T}$-spectra that commutes with the structure map.

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A weak equivalence of pre-cyclotomic spectra is a map that is an $\mathcal{F}_p$-equivalence of the underlying orthogonal $\mathbb{T}$-spectra.

This is precisely a map that is a weak equivalence of non-equivariant spectra on C_{p^m} geometric fixed points for all $m \geq 0$.

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$$\mathbb{F}X = X \vee \rho^*\Phi^{C_p}X \vee \rho^*\Phi^{C_p}(\rho^*\Phi^{C_p}X) \vee \rho^*\Phi^{C_p}(\rho^*\Phi^{C_p}(\rho^*\Phi^{C_p}X)) \vee \cdots$$

is a monad on the category of orthogonal $\mathbb{T}$-spectra.

FFX → FX
FX → X

The category of pre-cyclotomic spectra is the category of algebras over the monad $\mathbb{F}$.

Model* structure on pre-cyclotomic spectra created by $\mathbb{F}$ from the $\mathcal{F}_p$-local model structure on orthogonal $\mathbb{T}$-spectra.

Translation

Cofibrations are built by attaching cells of the form

$$\mathbb{F}\Sigma^\infty_V(S^{n-1} \times \mathbb{T}/C_{p^m})_+ \to \mathbb{F}\Sigma^\infty_V(B^{n-1} \times \mathbb{T}/C_{p^m})_+$$

Fibrations are fibrations of the underlying orthogonal $\mathbb{T}$-spectra

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Mapping Spectra

Do (pre-)cyclotomic spectra have mapping spectra?

Is Φ^{C_p} a spectral functor: $F^{\mathbb{T}}(T,U) \to F^{\mathbb{T}}(\Phi^{C_p}T, \Phi^{C_p}U)$?

Spectrum of maps is an equalizer

$$F_{Cy}(T,U) \to F^{\mathbb{T}}(T,U) \rightrightarrows F^{\mathbb{T}}(\rho^*\Phi^{C_p}T, U)$$

$$\begin{array}{ccc} \rho^*\Phi^{C_p}T & \longrightarrow & \rho^*\Phi^{C_p}U \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

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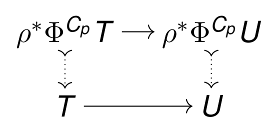

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Structure map commuting up to homotopy is a homotopy equalizer

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Theorem

If T is a cofibrant cyclotomic or pre-cyclotomic spectrum then

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Calculating Mapping Spectra

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TR is corepresentable

Let $S_{TR} = \Sigma^{\infty}_{\mathbb{T}}(\mathbb{T} \amalg (\mathbb{T}/C_p) \amalg (\mathbb{T}/C_{p^2}) \amalg \cdots)_+$

$$\Phi^{C_p}\Sigma^{\infty}_{\mathbb{T}}(\mathbb{T}/C_{p^m})_+ = \Sigma^{\infty}_{\mathbb{T}/C_p}(\mathbb{T}/C_{p^m})_+$$

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Quick Computation 1

$\rho^*\Phi^{C_p} S_{TR} = S_{TR}$

Use this for cyclotomic structure for S_{TR}.

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$$F^{\mathbb{T}}(S_{TR}, X) = X \times X^{C_p} \times X^{C_{p^2}} \times \cdots$$

$$\begin{array}{ccc} S_{TR} & \longrightarrow & \rho^* \Phi^{C_p} T \\ = \downarrow & & \downarrow \\ S_{TR} & \longrightarrow & T \end{array}$$

Conclusion

$$\begin{aligned} \mathbb{R}F_{Cy}(S_{TR}, T) &= (T \times T^{C_p} \times T^{C_{p^2}} \times \cdots)^{hR} \\ &= \text{holim}(\cdots \to T^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} T) \\ &= TR(T) \end{aligned}$$

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$$\begin{array}{ccc} S_{TR} & \longrightarrow & \rho^* \Phi^{C_p} T \\ \Big\downarrow = & & \Big\downarrow \\ S_{TR} & \longrightarrow & T \end{array}$$

Conclusion

$$\mathbb{R}F_{Cy}(S_{TR}, T) = (T \times T^{C_p} \times T^{C_{p^2}} \times \cdots)^{hR}$$

$$= \operatorname{holim}(\cdots \to T^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} T)$$

$$= TR(T)$$

TR is corepresentable

$$S_{TR} = \Sigma^{\infty}_{\mathbb{T}}(\mathbb{T} \amalg (\mathbb{T}/C_p) \amalg (\mathbb{T}/C_{p^2}) \amalg \cdots)_+$$

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TC is corepresentable (in pre-cyclotomic spectra)

Let $S_{TC,m} = \Sigma^{\infty}_{\mathbb{T}}(\mathbb{T}/C_{p^m})_+$ with pre-cyclotomic structure

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$F^{\mathbb{T}}(S_{TC,m}, X) = X^{C_{p^m}}$

Quick Computation

$\mathbb{R}F_{Cy}(S_{TC,m}, T)$ is the homotopy equalizer of $R, F \colon T^{C_{p^m}} \to T^{C_{p^{m-1}}}$.

Let $S_{TC} = \operatorname{hocolim} S_{TC_m}$

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Conjecture (Kaledin ICM 2010)

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