E_n Genera

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Equivariant, Chromatic, and Motivic Homotopy Theory

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Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Builds on Greg's thesis:
  Structured orientations of Thom spectra
  (Thesis, Indiana University, 2012)

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1. Introduction and main result

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3. Multiplicative Thom isomorphism
4. Topological Quillen cohomology and unstable obstructed Atiyah-Hirzebruch spectral sequences

Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of an abelian group A

$$M^m \mapsto \gamma(M) \in A$$

such that when M is a boundary (with extra structure) $\gamma(M) = 0$.

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A *genus* is a cobordism invariant for manifolds with extra structure:

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or, better, a map of ring spectra

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Structured Genera

Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

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MSO and *MU* are commutative *S*-algebras (E_∞ ring spectra).

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Which genera come from maps of "highly structured" ring spectra?

MSO and MU are commutative S-algebras (E_∞ ring spectra).

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- Maps of commutative S-algebras = E_∞ ring spectra
- Maps of S-algebras = A_∞ ring spectra = E_1 ring spectra
- Maps of E_n ring spectra ?

Main Result

Theorem

Let R be an E_2 ring spectrum with $\pi_n R = 0$ for all n odd.

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Consequence: Taking $R = MU$, the Quillen idempotent is an E_2 map, and BP is an E_2 ring spectrum (Chadwick's IU PhD Thesis) ⟵

Independent of [Basterra-Mandell] which showed that BP is E_4 but left open the question of Quillen idempotent

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Proof: Calculate π_* of the space of E_2 maps and look at the map to π_* of the space of ring spectra maps.

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Proof: Calculate π_0 of the space of E_2 maps and look at the map to π_0 of the space of ring spectra maps.

Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations

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An R-orientation for $\mathbb{C}^n$-vector bundles is an element of $R^{2n}(EU(n), \mathring{E}U(n))$ that restricts to a generator of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$ on each fiber (of $BU(n)$).

Multiplicative:

- Restricts to unit element of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$
- $$R^{2m}(EU(m), \mathring{E}U(m)) \quad \otimes \quad R^{2n}(EU(n), \mathring{E}U(n))$$ $$\to R^{2m+2n}((EU(m), \mathring{E}U(m)) \times (EU(n), \mathring{E}U(n)))$$ $$\to R^{2m+2n}(EU(m+n), \mathring{E}U(m+n))$$

Excision: $R^{2n}(EU(n), \mathring{E}U(n)) \cong \tilde{R}^{2n}(TU(n))$.

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The Thom Isomorphism

Thom diagonal

$$TU(n) \to TU \wedge BU(n)$$

$$MU \to MU \wedge BU_+$$

Gives an action

$$R^*(MU) \otimes R^*(BU) \to R^*(MU)$$

$$\left.\begin{aligned} f\colon MU \to R \\ g\colon \Sigma^\infty_+ BU \to R \end{aligned}\right\} \implies MU \to MU \wedge BU_+ \xrightarrow{f \wedge g} R \wedge R \to R$$

Taking f to be a fixed orientation, get a map

$$R^* BU \to R^* MU$$

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Multiplicative Structure

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is a map of ring spectra [Mahowald]

$$\begin{array}{ccccc}
MU \wedge MU & \longrightarrow & (MU \wedge BU_+) \wedge (MU \wedge BU_+) & \xrightarrow{\cong} & MU \wedge MU \wedge (BU \times BU)_+ \\
\downarrow & & & & \downarrow \\
MU & & \longrightarrow & & MU \wedge BU_+
\end{array}$$

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$$\begin{array}{ccc} MU \wedge MU \longrightarrow (MU \wedge BU_+) \wedge (MU \wedge BU_+) & \xrightarrow{\cong} & MU \wedge MU \wedge (BU \times BU)_+ \\ \downarrow & & \downarrow \\ MU & \longrightarrow & MU \wedge BU_+ \end{array}$$

Multiplicative Structure

Thom diagonal

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Multiplicative Orientations

For a multiplicative orientation σ

Thom map

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$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

takes

- Ring spectra maps $\Sigma^\infty_+ BU \to R$ = H-space maps $BU \to \Omega^\infty R^\times$ $\Big\}$ in $R^0(BU)$

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For a multiplicative orientation, the Thom map takes maps of H-spaces in $[BU, \Omega^\infty R^\times] = R^0(BU)$ to maps of ring spectra in $[MU, R] = R^0(MU)$

Theorem (Quillen)

Maps of ring spectra in $R^0(MU)$ are in one-to-one correspondence with elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$

When a map of ring spectra $MU \to R$ exists, then:

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And, well, you know, something about formal group laws

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E_n genera

Assume that R is E_n and $\sigma\colon MU \to R$ is E_n

Thom map

$$g\colon \Sigma^\infty_+ BU \to R \qquad \Longrightarrow \qquad MU \to MU \wedge BU_+ \xrightarrow{\sigma\wedge g} R \wedge R \to R$$

Fact:

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If R is E_{n+1} and then we have a natural action of the space of E_n maps $\mathcal{E}_n(BU, \Omega^\infty R^\times)$ on the space of E_n ring maps $\mathcal{E}_n(MU, R)$.

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This map is a weak equivalence.

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Let R be a connective E_n ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2\rangle \to R\langle 1\rangle \to R\langle 0\rangle = HZ$$

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Magic Fact (Kriz)

This is a tower of principal fibrations of E_n ring spectra

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Obstruction theory

- Obstruction in $H^{j+2}(A; \pi_{j+1} R)$ to lifting an E_n ring map $A \to R\langle j\rangle$ to an E_n ring map $A \to R\langle j+1\rangle$
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Atiyah-Hirzebruch Spectral Sequence

For E_n ring spectra A, R (with mild hypotheses on A), there is a natural “obstructed” spectral sequence

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Thom isomorphism: $HZ \wedge MU \xrightarrow{\simeq} HZ \wedge BU_+$ as E_n HZ-algebras.

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$$H^p_{\mathcal{E}_n}(\Sigma^\infty_+; \pi_q R) \xrightarrow{\cong} H^p_{\mathcal{E}_n}(MU; \pi_q R)$$

and an isomorphism

$$\pi_* \mathcal{E}_n(\Sigma^\infty_+ BU, R) \xrightarrow{\cong} \pi_* \mathcal{E}_n(MU, R)$$

Nothing special about BU/MU here; works for any E_n Thom spectrum.

Multiplicative Thom Isomorphism

Thom isomorphism: $HZ \wedge MU \xrightarrow{\simeq} HZ \wedge BU_+$ as E_n HZ-algebras.

$$HZ \wedge MU \to HZ \wedge MU \wedge BU_+ \to HZ \wedge HZ \wedge BU_+ \to HZ \wedge BU_+$$

$$\Longrightarrow \qquad H^*_{\mathcal{E}_n}(\Sigma^\infty_+ BU; -) \xrightarrow{\simeq} H^*_{\mathcal{E}_n}(MU; -)$$

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Computation

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\begin{aligned}
\mathcal{E}_n(\Sigma^\infty_+ BU, R) &\simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \\
&= \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \\
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Compute using "Atiyah-Hirzebruch spectral sequence"

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H^p(B^n BU; \bar{R}_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \\
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For $n = 2$,

$$H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots]$$

and

$$H^*(BSU) \to H^*(\Sigma^2 BU(1))$$

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Main Result

Theorem

If R is an even E_3 ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-*}\mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of E_2 ring spectra.

What about for R just E_2?

$$H^*_{\mathcal{E}_2}(MU;\pi) \cong H^*_{\mathcal{E}_2}(\Sigma^\infty_+ BU;\pi) \cong H^{*+2}(B^nBU;\pi)$$
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Careful argument with "Atiyah-Hirzebruch spectral sequence"

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