$E_n$ Genera

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Overview

Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
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1. Introduction and main result
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Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Builds on Greg’s thesis:  
  *Structured orientations of Thom spectra*  
  (Thesis, Indiana University, 2012)

Outline

1. Introduction and main result
2. Genera and orientations
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1. Introduction and main result
2. Genera and orientations
3. Multiplicative Thom isomorphism
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2. Genera and orientations
3. Multiplicative Thom isomorphism
4. Topological Quillen cohomology
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Which cobordism invariants are realized as maps of highly structured ring spectra?

- Joint work with Greg Chadwick (UC Riverside)
- Builds on Greg’s thesis: Structured orientations of Thom spectra
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Outline

1. Introduction and main result
2. Genera and orientations
3. Multiplicative Thom isomorphism
4. Topological Quillen cohomology and unstable obstructed Atiyah-Hirzebruch spectral sequences
A genus is a cobordism invariant for manifolds with extra structure: It assigns to every manifold (with extra structure) an element of an abelian group \( A \)

\[ M^m \mapsto \gamma(M) \in A \]

such that when \( M \) is a boundary (with extra structure) \( \gamma(M) = 0 \).
Review of Genera

Definition

A genus is a cobordism invariant for manifolds with extra structure:
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Extra structure: (e.g.)
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- Oriented manifolds ($MSO_*$)
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Definition

A *genus* is a cobordism invariant for manifolds with extra structure: It assigns to every manifold (with extra structure) an element of a graded abelian group $A_*$

$$M^m \mapsto \gamma(M) \in A_m$$

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Extra structure: (e.g.)

- Oriented manifolds ($MSO_*$)
- Stably almost complex manifolds ($MU_*$)
A *genus* is a cobordism invariant for manifolds with extra structure:
It assigns to every manifold (with extra structure) an element of a graded ring $R_*$

$$M^m \mapsto \gamma(M) \in R_m$$

such that when $M$ is a boundary (with extra structure) $\gamma(M) = 0$.

Extra structure: (e.g.)
- Oriented manifolds ($\text{MSO}_*$)
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Definition

A genus is a cobordism invariant for manifolds with extra structure:
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\[
M^m \mapsto \gamma(M) \in R_m
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such that when \( M \) is a boundary (with extra structure) \( \gamma(M) = 0 \) and for any \( M^m, N^n \),

\[
\gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n}
\]

Extra structure: (e.g.)
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Extra structure: (e.g.)

- Oriented manifolds with map to \( X \) \( (MSO_*(X)) \)
- Stably almost complex manifolds with map to \( X \) \( (MU_*(X)) \)
Review of Genera

Definition

A *genus* is a cobordism invariant for manifolds with extra structure:

It assigns to every manifold (with extra structure) an element of homology theory $R^*$

$$ M^m \mapsto \gamma(M) \in R_m(X) $$

such that when $M$ is a boundary (with extra structure) $\gamma(M) = 0$ and for any $M^m, N^n$,

$$ \gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n} $$

Extra structure: (e.g.)

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Review of Genera

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A genus is a cobordism invariant for manifolds with extra structure:

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$$M^m \mapsto \gamma(M) \in R_m(X)$$

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$$\gamma(M \times N) = \gamma(M) \cdot \gamma(N) \in R_{m+n}(X \times Y)$$

Extra structure: (e.g.)

- Oriented manifolds with map to $X$ \((MSO_*(X))\)
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A genus is a map of multiplicative homology theories

\[ MSO_\ast \to R_\ast \quad \text{or} \quad MU_\ast \to R_\ast \]

Extra structure: (e.g.)

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A genus is a map of multiplicative homology theories:

\[ MSO_* \rightarrow R_* \quad \text{or} \quad MU_* \rightarrow R_* \]

or, better, a map of ring spectra:

\[ MSO \rightarrow R \quad \text{or} \quad MU \rightarrow R \]

Extra structure: (e.g.)

- Oriented manifolds with map to \( X \) \((MSO_*(X))\)
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Structured Genera

Genera are maps of ring spectra $MSO \rightarrow R$, $MU \rightarrow R$. 

Which genera come from maps of "highly structured" ring spectra?

$MSO$ and $MU$ are commutative $S$-algebras ($E_{\infty}$ ring spectra).

Which genera come from maps of commutative $S$-algebras = $E_{\infty}$ ring spectra?

Maps of $E_n$ ring spectra?
Genera are maps of ring spectra $MSO \to R$, $MU \to R$.

**Question**
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$\text{MSO}$ and $\text{MU}$ are commutative $S$-algebras ($E_\infty$ ring spectra).

Which genera come from

- Maps of commutative $S$-algebras = $E_\infty$ ring spectra
- Maps of $S$-algebras = $A_\infty$ ring spectra = $E_1$ ring spectra ?
Genera are maps of ring spectra $MSO \rightarrow R$, $MU \rightarrow R$.

**Question**

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$MSO$ and $MU$ are commutative $S$-algebras ($E_\infty$ ring spectra).

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- Maps of commutative $S$-algebras = $E_\infty$ ring spectra
- Maps of $S$-algebras = $A_\infty$ ring spectra = $E_1$ ring spectra
- Maps of $E_n$ ring spectra

?
Main Result

Theorem

Let $R$ be an $E_2$ ring spectrum with $\pi_n R = 0$ for all $n$ odd.
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Let $R$ be an $E_2$ ring spectrum with $\pi_n R = 0$ for all $n$ odd.

Then any map of ring spectra $MU \to R$ lifts to a map of $E_2$ ring spectra.
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Then any map of ring spectra $MU \to R$ lifts to a map of $E_2$ ring spectra.

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Consequence: Taking $R = MU$, the Quillen idempotent is an $E_2$ map, and $BP$ is an $E_2$ ring spectrum (Chadwick’s IU PhD Thesis)

Independent of [Basterra-Mandell] which showed that $BP$ is $E_4$ but left open the question of Quillen idempotent
Main Result

Theorem

Let $R$ be an $E_\infty$ ring spectrum with $\pi_n R = 0$ for all $n$ odd. If there exists an $E_\infty$ ring map $\text{MU} \to R$ then any map of ring spectra $\text{MU} \to R$ lifts to a map of $E_2$ ring spectra.

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Consequence: Under the hypotheses above any map of ring spectra lifts to a map of $S$-algebras ($A_\infty$ ring spectra).
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Proof: Calculate $\pi_*$ of the space of $E_2$ maps and look at the map to $\pi_*$ of the space of ring spectra maps.
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Genera and Orientations

Pontryagin-Thom theorem

- Cobordism theories are represented by Thom spectra
- Genera are multiplicative orientations
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\[ BU(n) \text{ classifying space for } \mathbb{C}^n\text{-vector bundles} \]
\[ PU(n) \text{ total space of universal principal bundle} \]
\[ \text{(free contractible CW } U(n)\text{-space)} \]
\[ EU(n) = PU(n) \times_{U(n)} \mathbb{C}^n \text{ total space of universal vector bundle} \]
\[ EU(n) = PU(n) \times_{U(n)} (\mathbb{C}^n \setminus \{0\}) \]
\[ TU(n) = PU(n) + \wedge_{U(n)} S^{2n} \text{ Thom space} \]
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\[
(U(n) \text{-space})
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\]

\[
\tilde{EU}(n) = PU(n) \times U(n) (\mathbb{C}^n - \{0\})
\]

\[
TU(n) = PU(n) + \wedge U(n) S^{2n} \text{ Thom space}
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Genera and Orientations

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An $R$-orientation for $\mathbb{C}^n$-vector bundles is an element of $R^{2n}(EU(n), \hat{EU}(n))$ that restricts to a generator of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$ on each fiber (of $BU(n)$).

Multiplicative:

- Restricts to unit element of $R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\})$

$$R^{2m}(EU(m), \hat{EU}(m)) \otimes R^{2n}(EU(n), \hat{EU}(n)) \rightarrow R^{2m+2n}(EU(m), \hat{EU}(m)) \times (EU(n), \hat{EU}(n)))$$

$$\rightarrow R^{2m+2n}(EU(m + n), \hat{EU}(m + n))$$

Excision: $R^{2n}(EU(n), \hat{EU}(n)) \cong \tilde{R}^{2n}(TU(n))$.

$$MU = \text{colim} \sum_{-2n}^\infty TU(n) \quad [MU, R] = R^0(MU) = \lim \tilde{R}^{2n}(TU(n))$$
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- Restricts to unit element of \( R^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\}) \)
- \( R^{2m}(EU(m), \tilde{EU}(m)) \otimes R^{2n}(EU(n), \tilde{EU}(n)) \rightarrow R^{2m+2n}((EU(m), \tilde{EU}(m)) \times (EU(n), \tilde{EU}(n))) \rightarrow R^{2m+2n}(EU(m+n), \tilde{EU}(m+n)) \)

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\[ EU(n) = PU(n) \times U(n) \mathbb{C}^n \] total space of universal vector bundle
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\[ TU(n) = PU(n) + \wedge U(n) S^{2n} \] Thom space

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  \[ \to R^{2m+2n}(EU(m), \hat{EU}(m)) \times (EU(n), \hat{EU}(n)) \]
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$MU = \text{colim} \sum_{-2n}^{\infty} TU(n)$

$[MU, R] = R^0(MU) = \lim \tilde{R}^{2n}(TU(n))$
The Thom Isomorphism

Thom diagonal

\[ \text{Thom diagonal} \]

\[ \text{MU} \to \text{MU} \wedge \text{BU}^+ \]

Gives an action

\[ R^*(\text{MU}) \otimes R^*(\text{BU}) \to R^*(\text{MU}) \]

\[ f: \text{MU} \to R \]
\[ g: \Sigma_+ \infty \text{BU} \to R \]

\[ \implies \quad \text{MU} \to \text{MU} \wedge \text{BU}^+ \xrightarrow{f \wedge g} R \wedge R \to R \]

Taking \( f \) to be a fixed orientation, get a map

\[ R^* \text{BU} \to R^* \text{MU} \]

Thom Isomorphism: This map is an isomorphism
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$$MU \to MU \wedge BU_+$$

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\[ \begin{cases} 
  f: MU \to R \\
  g: \Sigma^\infty BU \to R 
\end{cases} \quad \Rightarrow \quad \begin{cases} 
  MU \to MU \wedge BU_+ \\
  R \wedge R \to R 
\end{cases} \]

Taking \( f \) to be a fixed orientation, get a map

\[ R^*BU \to R^*MU \]

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\[ \text{MU} \rightarrow \text{MU} \land \text{BU}_+ \]

Gives an action

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\[
\begin{align*}
  f &: \text{MU} \rightarrow R \\
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\end{align*}
\]

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Taking \( f \) to be a fixed orientation, get a map

\[ R^* \text{BU} \rightarrow R^* \text{MU} \]

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Thom diagonal

\[ MU \to MU \wedge BU_+ \]

is a map of ring spectra [Mahowald]

\[
\begin{array}{c}
MU \wedge MU \\
\downarrow
\end{array} \quad \begin{array}{c}
(MU \wedge BU_+) \wedge (MU \wedge BU_+)
\end{array} \xrightarrow{\sim} \begin{array}{c}
MU \wedge MU \wedge (BU \times BU)_+
\end{array} \quad \begin{array}{c}
\downarrow
\end{array}
\]

\[
\begin{array}{c}
MU
\end{array} \quad \begin{array}{c}
\rightarrow
\end{array} \quad \begin{array}{c}
MU \wedge BU_+
\end{array}
\]

[Image 349x19 to 360x33]
Thom diagonal

\[ \text{MU} \to \text{MU} \wedge \text{BU}_+ \]

is a map of ring spectra [Mahowald]

\[ \text{MU} \wedge \text{MU} \to (\text{MU} \wedge \text{BU}_+) \wedge (\text{MU} \wedge \text{BU}_+) \overset{\text{IR}}{\to} \text{MU} \wedge \text{MU} \wedge (\text{BU} \times \text{BU})_+ \]

\[ \text{MU} \to \text{MU} \wedge \text{BU}_+ \]
Thom diagonal

\[ MU \to MU \wedge BU_+ \]

is a map of ring spectra [Mahowald] in fact \( E_\infty \) ring spectra [Lewis]

\[
\begin{array}{c}
\mu \wedge \mu \to (\mu \wedge BU_+) \wedge (\mu \wedge BU_+) \\
\downarrow \quad \downarrow \\
\mu \quad \mu \wedge BU_+ \\
\end{array}
\]

\[
\begin{array}{c}
\mu \wedge \mu \wedge (BU \times BU)_+ \\
\downarrow \quad \downarrow \\
\mu \wedge BU_+ \\
\end{array}
\]
Multiplicative Orientations

For a multiplicative orientation $\sigma$

Thom map

$$g : \Sigma^\infty BU \to R \quad \implies \quad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

takes

- Ring spectra maps $\Sigma^\infty BU \to R$ = $H$-space maps $BU \to \Omega^\infty R^\times$ in $R^0(BU)$

to

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For a multiplicative orientation, the Thom map takes maps of $H$-spaces in $[BU, \Omega^\infty R^\times] = R^0(BU)$ to maps of ring spectra in $[MU, R] = R^0(MU)$

Theorem (Quillen)

Maps of ring spectra in $R^0(MU)$ are in one-to-one correspondence with elements of $\tilde{R}^2(TU(1))$ that restrict to the unit element of $\tilde{R}^2(S^2)$

When a map of ring spectra $MU \to R$ exists, then:

- The maps of ring spectra in $R^0(MU)$ are exactly the maps that correspond to maps of $H$-spaces in $R^0(BU)$ and
- Are in one-to-one correspondence with elements of $R^0(BU(1))$ via the map on $R^0$ induced by $BU(1) \to BU$. 
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And, well, you know, something about formal group laws
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For a multiplicative orientation, the Thom map takes maps of \( H \)-spaces in \([BU, \Omega^\infty R^\times]\) to maps of ring spectra in \([MU, R] = R^0(MU)\)
Assume that $R$ is $E_n$ and $\sigma : MU \to R$ is $E_n$.

Thom map

$$g : \Sigma_+ \infty BU \to R \quad \Rightarrow \quad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

Fact:

- Space of $E_n$ ring maps $\Sigma_+ \infty BU \to R$ isomorphic to space of $E_n$ maps $BU \to \Omega \infty R^\times$ [May-Quinn-Ray-Tornehave]
- If $R$ is $E_n$ then $R \wedge R \to R$ is $E_{n-1}$ [Dunn]
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Observation
If $R$ is $E_{n+1}$ and then we have a natural action of the space of $E_n$ maps $\mathcal{E}_n(BU, \Omega^\infty R^\times)$ on the space of $E_n$ ring maps $\mathcal{E}_n(MU, R)$. If non empty, we get a map $\mathcal{E}_n(BU, \Omega^\infty R^\times) \to \mathcal{E}_n(MU, R)$. 
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$$g : \Sigma_+ \infty BU \to R \quad \implies \quad MU \to MU \wedge BU_+ \xrightarrow{\sigma \wedge g} R \wedge R \to R$$

induces a map $\mathcal{E}_n(BU, \Omega_0 R^\times) \approx \mathcal{E}_n(\Sigma_+ \infty BU, R) \to \mathcal{E}_n(MU, R)$

**Theorem**

*This map is a weak equivalence.*

- Suffices to consider the case when $R$ is connective
- Look at Postnikov tower of $R$
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Postnikov Towers of $E_n$ ring spectra

Let $R$ be a connective $E_n$ ring spectrum and let $Z = \pi_0 R$.

Form Postnikov tower by killing higher homotopy groups

$$R \to \cdots \to R\langle 2 \rangle \to R\langle 1 \rangle \to R\langle 0 \rangle = HZ$$

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Magic Fact (Kriz)

This is a tower of principal fibrations of $E_n$ ring spectra

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$$R\langle j \rangle \to HZ \vee \Sigma^{j+2} H\pi_{j+1} R$$
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Topological Quillen Cohomology

\[
\begin{array}{ccc}
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\downarrow & & \downarrow \\
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\end{array}
\]
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\[ H^*_\mathcal{E}_n(A; M) := \pi_0\mathcal{E}_n/\mathcal{H}Z(A, \mathcal{H}Z \vee \Sigma^*\mathcal{H}M) \]

Obstruction theory

- Obstruction in \( H^{j+2}(A; \pi_{j+1} R) \) to lifting an \( E_n \) ring map \( A \to R\langle j \rangle \) to an \( E_n \) ring map \( A \to R\langle j + 1 \rangle \)
- The space of lifts is either empty or is a “free orbit” on the grouplike topological monoid

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Atiyah-Hirzebruch Spectral Sequence

For \( E_n \) ring spectra \( A, R \) (with mild hypotheses on \( A \)), there is a natural “obstructed” spectral sequence

\[ E^2_{p,q} = H^p_{\mathcal{E}_n}(A; \pi_q R) \implies \pi_{q-p}\mathcal{E}_n(A, R) \]
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M.A. Mandell (IU)
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Thom isomorphism: $HZ \wedge MU \xrightarrow{\sim} HZ \wedge BU_+$ as $E_n$ $HZ$-algebras.

For $R$ an $E_{n+1}$ ring spectrum and $\sigma: MU \to R$ an $E_n$ ring map, the Thom map induces an isomorphism on $E^2$-terms

$$H^p_{E_n}(\Sigma^\infty_+ BU; \pi_q R) \xrightarrow{\sim} H^p_{E_n}(MU; \pi_q R)$$

and an isomorphism

$$\pi_* E_n(\Sigma^\infty_+ BU, R) \xrightarrow{\sim} \pi_* E_n(MU, R)$$

Nothing special about $BU/MU$ here; works for any $E_n$ Thom spectrum.
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Thom isomorphism: \( HZ \wedge MU \xrightarrow{\sim} HZ \wedge BU_+ \) as \( E_n \) \( HZ \)-algebras.

\[
HZ \wedge MU \rightarrow HZ \wedge MU \wedge BU_+ \rightarrow HZ \wedge HZ \wedge BU_+ \rightarrow HZ \wedge BU_+ \\
\Rightarrow H^*_E(\Sigma^\infty BU; -) \xrightarrow{\sim} H^*_E(MU; -)
\]

Consequence

For \( R \) an \( E_{n+1} \) ring spectrum and \( \sigma: MU \rightarrow R \) an \( E_n \) ring map, the Thom map induces an isomorphism on \( E^2 \)-terms

\[
\xymatrix{ 
H^p_E(\Sigma^\infty BU; \pi_q R) 
\ar@{^{(}->}[r]^-{BU} 
\ar@{<->}[d]^-{\sim} 
& 
H^p_E(MU; \pi_q R) 
\ar@{<->}[d]^-{\sim} 
}
\]

and an isomorphism

\[
\pi_* E_n(\Sigma^\infty BU, R) \xrightarrow{\sim} \pi_* E_n(MU, R)
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$$HZ \wedge MU \rightarrow HZ \wedge MU \wedge BU_+ \rightarrow HZ \wedge HZ \wedge BU_+ \rightarrow HZ \wedge BU_+$$

$$\implies H_{E_n}^* (\Sigma\infty BU; -) \xrightarrow{\sim} H_{E_n}^* (MU; -)$$

Consequence

For $R$ an $E_{n+1}$ ring spectrum and $\sigma: MU \rightarrow R$ an $E_n$ ring map, the Thom map induces an isomorphism on $E^2$-terms

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Nothing special about \( BU/MU \) here; works for any \( E_n \) Thom spectrum.
\[ \mathcal{E}_n(\Sigma^\infty BU, R) \simeq \mathcal{E}_n(BU, \Omega^\infty R^\times) \]
\[ = \mathcal{E}_n(BU, (\Omega^\infty R^\times)_1) \]
\[ \simeq \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1) \]

Compute using “Atiyah-Hirzebruch spectral sequence”

\[ H^p(B^n BU; \bar{R}_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega^\infty R)_1)) \]
\[ \implies \pi_{q+n-p} \mathcal{U}(B^n BU, B^n(\Omega^\infty R)_1) \]

For \( n = 2 \),

\[ H^*(B^2 BU) = H^*(BSU) = \mathbb{Z}[c_2, c_3, \ldots] \]

and

\[ H^*(BSU) \to H^*(\Sigma^2 BU(1)) \]

is surjective.
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Compute using “Atiyah-Hirzebruch spectral sequence”

\[ H^p(B^n BU; \bar{R}_q) = H^p(B^n BU; \pi_{q+n}(B^n(\Omega \infty R)_1)) \]
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For \( n = 2 \),
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Theorem

If $R$ is an even $E_3$ ring spectrum, then $\mathcal{E}_2(MU, R) \simeq \mathcal{E}_2(BU, \Omega^\infty R^\times)$ and $\pi_{-k} \mathcal{E}_2(BU, \Omega^\infty R^\times) \to R^*(BU(1))$ is surjective. Thus, every map of ring spectra $MU \to R$ lifts to a map of $E_2$ ring spectra.

What about for $R$ just $E_2$?
\[
H^*_\mathcal{E}_2(MU; \pi) \cong H^*_\mathcal{E}_2(\Sigma^\infty_+ BU; \pi) \cong H^*+2(B^n BU; \pi) = H^*+2(BSU; \pi) \to H^*(BU(1); \pi)
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Careful argument with “Atiyah-Hirzebruch spectral sequence”

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