A Strong Künneth Theorem for Periodic Topological Cyclic Homology

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Indiana University

Shanks Workshop on Homotopy Theory

Vanderbilt University

March 25, 2017

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Overview

Topological periodic cyclic homology (*TP*) is the analogue of periodic cyclic homology (*HP*) using *THH* in place of *HH*. If k is a finite field, then smooth and proper d.g. categories over k satisfy a strong Künneth theorem:

$$TP(X) \wedge^{L}_{TP(k)} TP(Y) \to TP(X \otimes_k Y)$$

is an isomorphism in the derived category of $TP(k)$-modules.

- Joint work with Andrew Blumberg
- Preprint Soon

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1. Introduction to *TP*

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1. Introduction to *TP*
2. Structure and properties of *TP*

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1. Introduction to *TP*
2. Structure and properties of *TP*
3. The Künneth theorem

Hochschild Homology

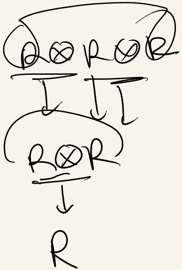
Cyclic bar construction

$$N_q^{cy} R = \underbrace{R \otimes \cdots \otimes R}_{q \text{ factors}} \otimes R$$

$$R \otimes \cdots \otimes R$$

$$\otimes \qquad\qquad \otimes$$

$$R$$



Chain complex

Cyclic structure $\Longrightarrow$ Connes' B operator $\quad B\colon N^{cy}R \to N^{cy}R[-1]$

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$B^2 = 0$

Hochschild Homology and Cyclic Homology

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$$\begin{array}{ccc} & R \otimes \cdots \otimes R & \\ \otimes & & \otimes \\ & R & \end{array}$$

Chain complex

Construct Double Complex:

$$\begin{array}{c} \vdots \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \end{array}$$

HH

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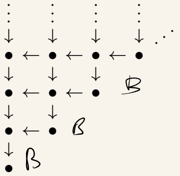

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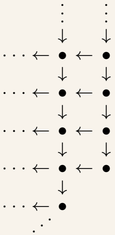

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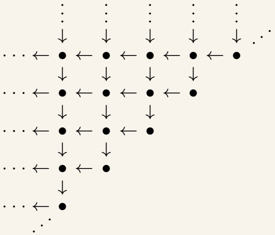

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Topological Hochschild Homology

Cyclic bar construction (Bökstedt)

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Spectrum

Cyclic structure $\Longrightarrow$ circle group action

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Spectrum

Cyclic structure $\Longrightarrow$ circle group action

$S^1_+ \simeq S^1 \vee S^0$

$\Sigma X \to S^1_+ \wedge X \to X$

$X \to \Omega X$

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Construction

$$\vdots$$
$$\downarrow$$
$$\bullet$$
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HH corresponds to *THH*

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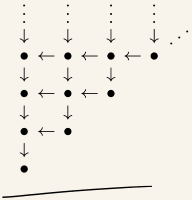

HH corresponds to *THH*
HC corresponds to $THH_{h\mathbb{T}}$

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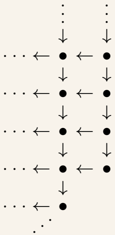

HH corresponds to THH
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Cyclic structure $\implies$ circle group action

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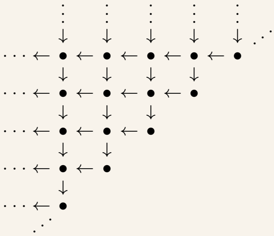

HH corresponds to *THH*
HP corresponds to $THH^{t\mathbb{T}}$

Topological Periodic Cyclic Homology

Definition

For a ring spectrum R, define the Topological Periodic Cyclic Homology of R by $TP(R) = THH(R)^{t\mathbb{T}}$.

Not always periodic e.g. $\pi_* TP(\mathbb{Z})$
not periodic

But $\pi_* TP(k)$ is period
$= Wk[v, v^{-1}]$ $|v| = -2$.

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Highlights

- Major player in trace method K-theory calculations
- Characteristic p replacement for HP (?)

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X smooth proper / $\mathbb{C}$

or smooth f.g. $\mathbb{C}$-alg

$HP_0(X)$ ⊕ de Rham cohomology

[illegible]

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Realization functor / Weil cohomology theory

$$HP_*(X) \otimes_{k[t,t^{-1}]} HP_*(Y) \to HP_*(X \otimes_k Y)$$

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Realization functor / Weil cohomology theory

$$HP_*(X) \otimes_{k[t,t^{-1}]} HP_*(Y) \to HP_*(X \otimes_k Y)$$

$|t| = 2$

Künneth Theorem

Theorem

Lax symmetric monoidal functor

$$TP(X) \wedge^{L}_{TP(R)} TP(Y) \to TP(X \wedge^{L}_{R} Y)$$

Bokstedt
McClure
⇒ Tate
THH
is Lax symm
mon.

Definition

A k-algebra X is smooth when it is compact as an $X \otimes_k X^{\mathrm{op}}$-module, i.e., when $R\,\mathrm{Hom}^{X \otimes_k X^{\mathrm{op}}}(X, -)$ commutes with direct sums.

Definition

A k-algebra X is proper when it is compact as a k-module.

Künneth Theorem

Theorem

Let k be finite field. The lax symmetric monoidal functor

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stable cat
derived cat of TP(k)-mod

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Review of Tate Construction

$E\mathbb{T}$ $\qquad E\mathbb{T}_+ \to S^0 \to \widetilde{E\mathbb{T}}$

Smash with $Z^{E\mathbb{T}}$ and take fixed points

$$(Z^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{\mathbb{T}} \to (Z^{E\mathbb{T}})^{\mathbb{T}} \to (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

$(X^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{\mathbb{T}} \simeq \Sigma(X^{E\mathbb{T}})_{h\mathbb{T}} \simeq \Sigma X h\mathbb{T}$ (Adams Isomorphism)

Definition

For Z a $\mathbb{T}$-equivariant spectrum $Z^{t\mathbb{T}} = (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$.
(Composite of derived functors.)

$$\Sigma Z_{h\mathbb{T}} \to Z^{h\mathbb{T}} \to Z^{t\mathbb{T}} \to \Sigma^2 Z_{h\mathbb{T}}$$

$$TP(X) = THH(X)^{t\mathbb{T}}$$

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$$\Sigma Z_{h\mathbb{T}} \to Z^{h\mathbb{T}} \to Z^{t\mathbb{T}} \to \Sigma^2 Z_{h\mathbb{T}}$$

$$TP(X) = THH(X)^{t\mathbb{T}}$$

Review of Tate Construction

$E\mathbb{T}$ $\qquad E\mathbb{T}_+ \to S^0 \to \widetilde{E\mathbb{T}}$

Smash with $Z^{E\mathbb{T}}$ and take fixed points

$$(Z^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{\mathbb{T}} \to (Z^{E\mathbb{T}})^{\mathbb{T}} \to (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

$(X^{E\mathbb{T}} \wedge E\mathbb{T}_+)^{\mathbb{T}} \simeq \Sigma(X^{E\mathbb{T}})_{h\mathbb{T}} \simeq \Sigma X h\mathbb{T}$ (Adams Isomorphism)

Definition

For Z a $\mathbb{T}$-equivariant spectrum $Z^{t\mathbb{T}} = (Z^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$.
(Composite of derived functors.)

$$\Sigma Z_{h\mathbb{T}} \to Z^{h\mathbb{T}} \to Z^{t\mathbb{T}} \to \Sigma^2 Z_{h\mathbb{T}}$$

$$TP(X) = THH(X)^{t\mathbb{T}}$$

The Multiplication

$$TP(X) \wedge TP(Y) \to TP(X \wedge Y)$$

$$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$$

- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$
- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$
- $THH(X) \wedge THH(Y) \cong THH(X \wedge Y)$

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$$TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$$

The Multiplication

$$TP(X) \wedge TP(Y) \to TP(X \wedge Y)$$

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$$TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$$

$$TP(X) \wedge TP(R) \wedge TP(Y) \to TP(X \wedge R \wedge Y)$$

The Multiplication

$$TP(X) \wedge TP(Y) \to TP(X \wedge Y)$$

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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$
- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$ $\longleftarrow$ This is coherent
- $THH(X) \wedge THH(Y) \cong THH(X \wedge Y)$

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- $\widetilde{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}} \simeq \widetilde{E\mathbb{T}}$ $\longleftarrow$ This can be made coherent!
- Use diagonal map $E\mathbb{T} \to E\mathbb{T} \times E\mathbb{T}$ $\longleftarrow$ This is coherent
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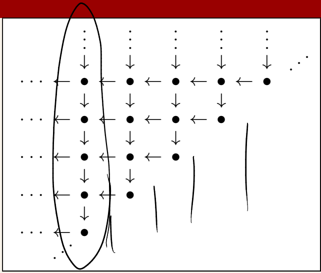
The Filtration



Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$

$TP(X) = (THH(X)^{E\mathbb{T}} \wedge \widetilde{E\mathbb{T}})^{\mathbb{T}}$

Simplicial filtration on $E\mathbb{T}$

$\mathbb{T}_+, \Sigma^2\mathbb{T}_+, \Sigma^4\mathbb{T}_+, \ldots$

The Filtration

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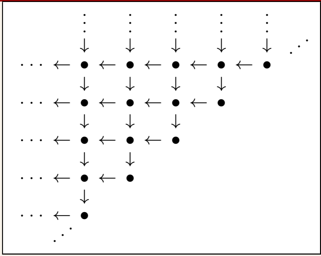

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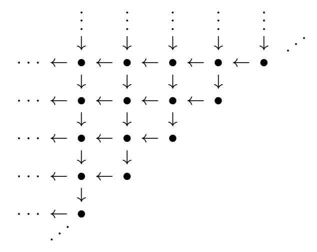
The Filtration



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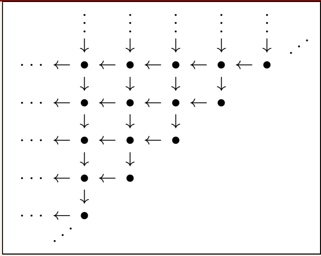

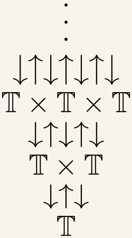

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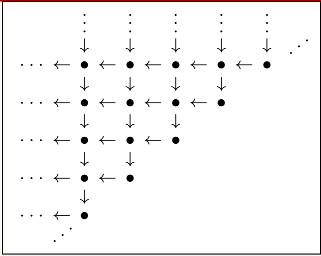

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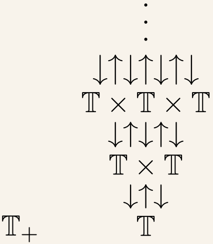

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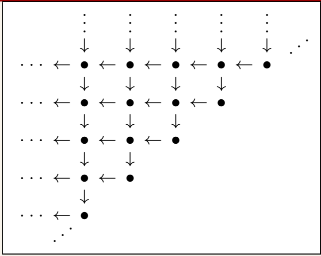

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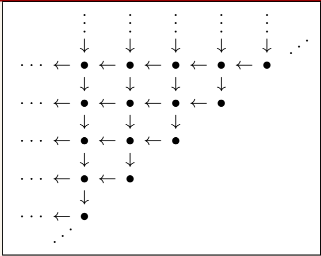
Simplicial filtration on $E\mathbb{T}$

$$\mathbb{T}_+,\ \Sigma^2\mathbb{T}_+,\ \Sigma^4\mathbb{T}_+, \dots$$

$$\mathbb{T}_+ \wedge (\mathbb{T}/\{1\}) \wedge \Delta[1]/\partial\Delta[1]$$

$$\begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ \mathbb{T}\times\mathbb{T}\times\mathbb{T} \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ \mathbb{T}\times\mathbb{T} \\ \downarrow\uparrow\downarrow \\ \mathbb{T} \end{array}$$

The Filtration



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$$\mathbb{T}_+ \wedge (\mathbb{T} \times \mathbb{T}/(\mathbb{T} \vee \mathbb{T})) \wedge \Delta[2]/\partial\Delta[2]$$

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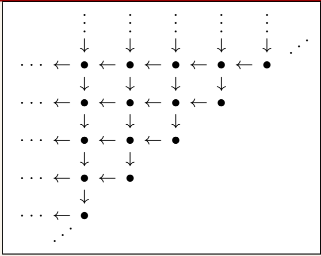

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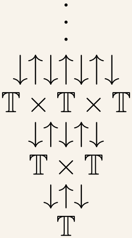

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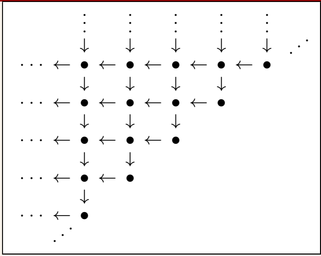

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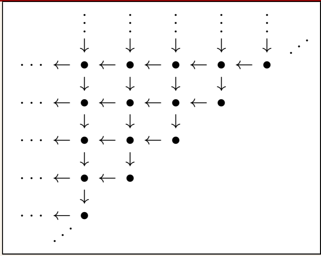
The Filtration



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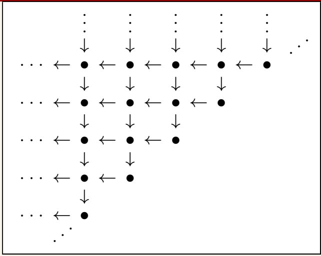
$$F^i/F^{i-1} = \begin{cases} (THH(X)^{(\Sigma^{2i}\mathbb{T}_+)})^{\mathbb{T}} & i \leq 0 \\ (THH(X)^{E\mathbb{T}} \wedge \Sigma^{2i-1}\mathbb{T}_+)^{\mathbb{T}} & i > 0 \end{cases}$$

The Filtration



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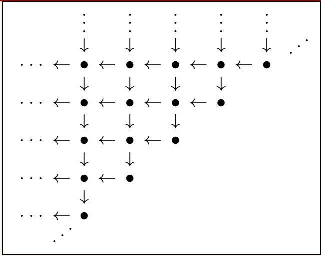

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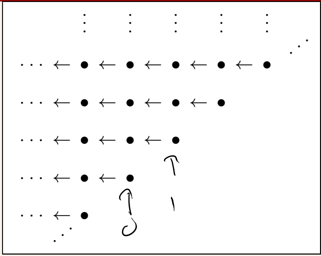
The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$

Spectral sequence

$$E^1_{i,j} = \pi_{i+j}\Sigma^{2i} THH(X) = THH_{j-i}(X)$$



$(1,1)$ periodic on E^1

Renumber: Double filtration degree

$$E^{2r}_{2i,j} = (E^r_{i,i+j})^{\text{old}}, \qquad d_{2r} = (d_r)^{\text{old}}$$

Spectral Sequence

Conditionally convergent spectral sequence

$$E^2_{2i,j} = THH_j(X) \Longrightarrow TP_{2i+j}(X). \qquad (E^r_{2i+1,j} = 0)$$

The Spectral Sequence

Filtration on $TP(X)$ with associated graded

$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$

Spectral sequence

$$E^1_{i,j} = \pi_{i+j}\Sigma^{2i} THH(X) = THH_{j-i}(X)$$

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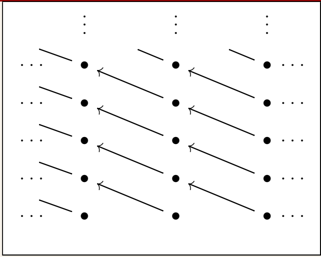

$(2,0)$ periodic on E^2

Spectral Sequence

Conditionally convergent spectral sequence

$$E^2_{2i,j} = THH_j(X) \Longrightarrow TP_{2i+j}(X). \qquad (E^r_{2i+1,j} = 0)$$

The Spectral Sequence

Filtration on $TP(X)$ with associated graded

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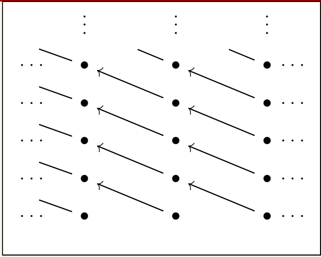


$(2,0)$ periodic on E^2

Spectral Sequence

Conditionally convergent spectral sequence

$$E^2_{2i,j} = THH_j(X) \Longrightarrow TP_{2i+j}(X). \qquad (E^r_{2i+1,j} = 0)$$

The Spectral Sequence

Filtration on $TP(X)$ with associated graded

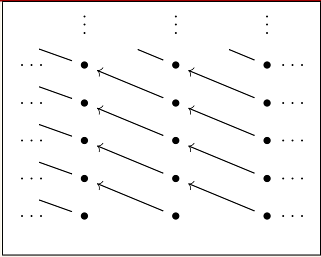
$$F^i/F^{i-1} \simeq \Sigma^{2i} THH(X)$$

Spectral sequence

$$E^1_{i,j} = \pi_{i+j}\Sigma^{2i} THH(X) = THH_{j-i}(X)$$

Renumber: Double filtration degree

$$E^{2r}_{2i,j} = (E^r_{i,i+j})^{\text{old}}, \qquad d_{2r} = (d_r)^{\text{old}}$$



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Approximation of the diagonal for simplicial spaces

Let $X_\bullet$ be a simplicial space, $|X_\bullet|$ its geometric realization. $|X^n_\bullet|$ vs $|X_\bullet|^n$

Problem

Parametrize a contractible spaces of filtered approximations of the diagonal maps $|X_\bullet| \to |X_\bullet|^n$ for all n that compose appropriately.

Find an A_∞ operad $\mathcal{A}$ and a map of operads

$$\mathcal{A}(n) \to \mathrm{Filt}(|X_\bullet|, |X_\bullet|^n) \subset \mathscr{T}(|X_\bullet|, |X_\bullet|^n).$$

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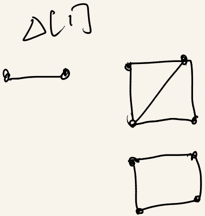

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Barycentric Coordinates and Milnor Coordinates on $\Delta[m]$

Barycentric $t_0, \dots, t_m$, $\sum t_i = 1 \leftrightarrow$ Milnor $0 \leq u_0 \leq u_1 \leq \cdots \leq u_{m-1} \leq 1$

An element in $|X_\bullet|$ is specified by $(x \in X_m, 0 \leq u_0 \leq \cdots \leq u_{m-1} \leq 1)$

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1. The overlapping little 1-cubes operad $\mathcal{C}_1^{\equiv}$

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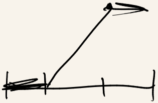

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Filtered Monoidal Structure

A filtered approximation of the diagonal gives a map

$$E\mathbb{T}_{i+j-1} \to (E\mathbb{T}_{i-1} \times E\mathbb{T}) \cup (E\mathbb{T} \times E\mathbb{T}_{j-1}) \subset E\mathbb{T} \times E\mathbb{T}$$

Hence a map $(E\mathbb{T}, E\mathbb{T}_{i+j-1}) \to (E\mathbb{T}, E\mathbb{T}_{i-1}) \times (E\mathbb{T}, E\mathbb{T}_{j-1})$

Applied to TP

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Little 1-cubes: Moore construction (Moore loop space)

Use length parameter to make fully associative

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Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$

$\implies$ map of spectral sequences

Righthand spectral sequence is Tate spectral sequence for

$THH(X \wedge_R Y) \cong THH(X) \wedge_{THH(R)} THH(Y)$

E^2 periodic with $\pi_*(THH(X) \wedge_{THH(R)} THH(Y))$ in each even column

Lefthand spectral sequence has (renumbered) E^2-term

$\pi_* \operatorname{Gr}(TP(X) \wedge_{TP(R)} TP(Y)) \cong \pi_*(\operatorname{Gr} TP(X) \wedge_{\operatorname{Gr} TP(R)} \operatorname{Gr} TP(Y))$

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E^2-term is $\pi_* \operatorname{Gr} TP(R)$-module $\Longrightarrow$ $(2,0)$-periodic

Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$

$\implies$ map of spectral sequences

Righthand spectral sequence is Tate spectral sequence for

$THH(X \wedge_R Y) \cong THH(X) \wedge_{THH(R)} THH(Y)$

E^2 periodic with $\pi_*(THH(X) \wedge_{THH(R)} THH(Y))$ in each even column

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Proposition

Map of spectral sequences is an isomorphism on E^2

Outline of Proof of Künneth Theorem

Filtered map $TP(X) \wedge_{TP(R)} TP(Y) \to TP(X \wedge_R Y)$
induces isomorphism of E^2-terms of spectral sequences

RHSS: Tate spectral sequence $\implies$ conditionally convergent.

Theorem

If $R = Hk$ and X and Y are smooth and proper, then the LHSS is conditionally convergent.

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