Localization Sequences in $THH$

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Topology Seminar
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Overview

Localization Sequences in $THH$

- Joint work with Andrew Blumberg
Overview

Localization Sequences in $THH$ and $TC$

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Localization Sequences in $THH$ and $TC$ II

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**Main Goal:** Prove Ausoni–Rognes/Hesselholt conjecture about the localization sequences for $THH(ku)$ (and $TC(ku)$)
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Localization Sequences in $THH$ and $TC$ II

- Joint work with Andrew Blumberg

**Main Goal:** Prove Ausoni–Rognes/Hesselholt conjecture about the localization sequences for $THH(ku)$ (and $TC(ku)$)

**2nd Goal:** Make sense of the $THH$ of Waldhausen categories
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Localization Sequences in $THH$ and $TC$ II

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**2nd Goal**: Make sense of the $THH$ of Waldhausen categories

**3rd Goal**: Understand the relationship to already known localization sequences in $THH$ and $TC$
Quillen Localization Sequence

Let $R$ be a discrete valuation ring, with residue field $k$ and field of fractions $F$. Then there is a cofibration sequence of $K$-theory spectra

$$K(k) \rightarrow K(R) \rightarrow K(F).$$

$k = R/\pi$

$F = R[\pi^{-1}]$

DVR = local ring

PID

$R = \mathbb{Z}_p$

$k = \mathbb{Z}/p$

$F = \mathbb{Q}_p$

PID will survive irreducible (up to unit)

$\pi$
Let $R$ be a discrete valuation ring, with residue field $k$ and field of fractions $F$. Then there is a cofibration sequence of $K$-theory spectra

$$K(k) \to K(R) \to K(F).$$

This uses the $K$-theory of abelian categories.

Secretly $K(k)$ is really the $K$-theory of the category of finitely generated torsion $R$-modules.

(Devissage theorem.)
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(Devissage theorem.)
Hesselholt–Madsen Theorem

Let $R$ be a complete discrete valuation ring, with residue field $k$ perfect of characteristic $p > 2$ and field of fractions $F$ of characteristic zero, containing the $p^n$-th roots of unity. Then:

- $K(F; \mathbb{Z}/p^n)$ can be computed in terms of the De Rham–Witt complex.
- $F$ satisfies the Lichtenbaum-Quillen conjecture. (proved by Voevodsky)


$$R = \varprojlim_n R/\pi^n$$
1. McCarthy Theorem: If $A \to B$ is nilpotent with surjective kernel

\[
\begin{array}{c}
K(A)_p \to K(B)_p \\
\downarrow \downarrow \\
TC(A) \to TC(B)
\end{array}
\]

is homotopy cartesian.

2. Quillen–Krasner Theorem: If $k$ is a perfect field of characteristic $p$, then $K(k)_p \cong H\mathbb{Z}_p$.

3. Suslin $K$-theory continuity results imply $K(R)_p \cong \text{holim} \ K(R/\pi^n)_p$.

Hesselholt–Madsen then show:

\[
\begin{align*}
K(k)_p & \cong TC(k)[0, \infty) \\
K(R)_p & \cong TC(R)[0, \infty)
\end{align*}
\]
1 McCarthy Theorem: If $A \to B$ is nilpotent with surjective kernel

$$K(A)_{\hat{\cdot}} \to K(B)_{\hat{\cdot}} \to TC(A) \to TC(B)$$

is homotopy cartesian.

2 Quillen–Krasner Theorem: If $k$ is a perfect field of characteristic $\rho$, then $K(k)_{\hat{\rho}} \simeq H\mathbb{Z}_{\hat{\rho}}$.

3 Suslin $K$-theory continuity results imply $K(R)_{\hat{\rho}} \simeq \text{holim} K(R/\pi^n)_{\hat{\rho}}$.

Hesselholt–Madsen then show:

$$K(k)_{\hat{\rho}} \simeq TC(k)[0, \infty) \quad K(R)_{\hat{\rho}} \simeq TC(R)[0, \infty)$$
1. McCarthy Theorem: If $A \to B$ is nilpotent with surjective kernel

\[ K(A)_{\hat{\rho}} \to K(B)_{\hat{\rho}} \to TC(A) \to TC(B) \]

is homotopy cartesian.

2. Quillen–Krasner Theorem: If $k$ is a perfect field of characteristic $p$, then

\[ K(k)_{\hat{\rho}} \cong H\mathbb{Z}_{\hat{\rho}}. \]

\[ b_2 \cong R/(\pi) \leftarrow R/(\pi^n) \]

3. Suslin $K$-theory continuity results imply $K(R)_{\hat{\rho}} \cong \text{holim} \ K(R/\pi^n)_{\hat{\rho}}$.

Hesselholt–Madsen then show:

\[ K(k)_{\hat{\rho}} \cong TC(k)[0, \infty) \]

\[ K(R)_{\hat{\rho}} \cong TC(R)[0, \infty) \]
1 McCarthy Theorem: If $A \to B$ is nilpotent with surjective kernel

\[
K(A)_{\hat{\cdot}} \to K(B)_{\hat{\cdot}}
\]

\[
\downarrow \quad \downarrow
\]

\[
TC(A) \to TC(B)
\]

is homotopy cartesian.

2 Quillen–Krasner Theorem: If $k$ is a perfect field of characteristic $p$, then $K(k)_{\hat{\cdot}} \simeq H\mathbb{Z}_{\hat{\cdot}}$. $R = \varprojlim R/\pi^n$

3 Suslin $K$-theory continuity results imply $K(R)_{\hat{\cdot}} \simeq \text{holim} K(R/\pi^n)_{\hat{\cdot}}$.

Hesselholt–Madsen then show:

\[
K(k)_{\hat{\cdot}} \simeq TC(k)[0, \infty)
\]

\[
K(R)_{\hat{\cdot}} \simeq TC(R)[0, \infty)
\]
1. McCarthy Theorem: If \( A \rightarrow B \) is nilpotent with surjective kernel

\[
\begin{align*}
K(A)_p & \rightarrow K(B)_p \\
\downarrow & \\
TC(A) & \rightarrow TC(B)
\end{align*}
\]
is homotopy cartesian.

2. Quillen–Krasner Theorem: If \( k \) is a perfect field of characteristic \( p \), then

\[
K(k)_p \simeq H\mathbb{Z}_p.
\]

3. Suslin \( K \)-theory continuity results imply

\[
K(R)_p \simeq \text{holim} K(R/\pi_1^n)_p.
\]

Hesselholt–Madsen then show:

\[
\begin{align*}
K(k)_p & \simeq TC(k)[0, \infty) \\
K(R)_p & \simeq TC(R)[0, \infty)
\end{align*}
\]
Key step to enable computation

Identify the cofiber of $TC(k) \to TC(R)$ in intrinsic terms.

Identify the cofiber of $THH(k) \to THH(R)$ in intrinsic terms.

Note: Cofiber is not $THH(F)$

Example

\[
R = \mathbb{Z}_p^\wedge, \quad k = \mathbb{Z}/p, \quad F = \mathbb{Q}_p^\wedge
\]

- $THH(\mathbb{Z}/p) = \vee \Sigma^{2n} H\mathbb{Z}/p$
- $THH(\mathbb{Z}_p^\wedge) = \vee \Sigma^{2n-1} H\mathbb{Z}_p^\wedge/n$
- $THH(\mathbb{Q}_p^\wedge) = H\mathbb{Q}_p^\wedge$

*Key step but not “big idea”. Big idea is the computation itself and interpretation in terms of De Rham–Witt.*
Key step to enable computation

Identify the cofiber of $TC(k) \rightarrow TC(R)$ in intrinsic terms.

Identify the cofiber of $THH(k) \rightarrow THH(R)$ in intrinsic terms.

Note: Cofiber is not $THH(F)$

**Example**

$R = \mathbb{Z}_p^\wedge$, $k = \mathbb{Z}/p$, $F = \mathbb{Q}_p^\wedge$

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- $THH(\mathbb{Q}_p^\wedge) = H\mathbb{Q}_p^\wedge$

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Identify the cofiber of $TC(k) \to TC(R)$ in intrinsic terms.

Identify the cofiber of $THH(k) \to THH(R)$ in intrinsic terms.

Note: Cofiber is not $THH(F)$

Example

\[
\begin{align*}
R &= \mathbb{Z}_p^\wedge, & k &= \mathbb{Z}/p, & F &= \mathbb{Q}_p^\wedge \\
& & & & \\
& & & & THH(\mathbb{Z}/p) &= \bigvee \Sigma^{2n} H\mathbb{Z}/p \\
& & & & THH(\mathbb{Z}_p^\wedge) &= \bigvee \Sigma^{2n-1} H\mathbb{Z}_p^\wedge/n \\
& & & & THH(\mathbb{Q}_p^\wedge) &= H\mathbb{Q}_p^\wedge
\end{align*}
\]

*Key step but not “big idea”. Big idea is the computation itself and interpretation in terms of De Rham–Witt.*
Key step to enable computation

Identify the cofiber of $TC(k) \to TC(R)$ in intrinsic terms.

Identify the cofiber of $THH(k) \to THH(R)$ in intrinsic terms.

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Example

$R = \mathbb{Z}_p^\wedge$, $k = \mathbb{Z}/p$, $F = \mathbb{Q}_p^\wedge$

- $THH(\mathbb{Z}/p) = \bigvee \Sigma^{2n} H\mathbb{Z}/p$
- $THH(\mathbb{Z}_p^\wedge) = \bigvee \Sigma^{2n-1} H\mathbb{Z}_p^\wedge / n$
- $THH(\mathbb{Q}_p^\wedge) = H\mathbb{Q}_p^\wedge$

*Key step but not “big idea”. Big idea is the computation itself and interpretation in terms of De Rham–Witt.*
Idea

Use Waldhausen’s localization sequence in $K$-theory to construct a localization sequence in $\text{THH}$.

$$\text{THH}(k) \to \text{THH}(R) \to \text{THH}(R \mid F)$$

Recall

$$\text{THH}(k) = \text{THH}(\mathcal{T}) = N^c_{\text{Bök}}(S \cdot \mathcal{T}) \simeq N^c_{\text{Bök}}(S \cdot N^i \mathcal{T})$$

$$\text{THH}(R) = \text{THH}(\mathcal{M}) = N^c_{\text{Bök}}(S \cdot \mathcal{M}) \simeq N^c_{\text{Bök}}(S \cdot N^i \mathcal{M})$$

$\mathcal{T} =$ Torsion f.g. $R$-modules

$\mathcal{M} =$ All f.g. $R$-modules
Idea

Use Waldhausen’s localization sequence in $K$-theory to construct a localization sequence in $THH$.

$$THH(k) \rightarrow THH(R) \rightarrow THH(R \mid F)$$

Recall

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$$THH(R) = THH(\mathcal{M}) = N^{\text{cyc}}_{\text{Bök}}(S \cdot \mathcal{M}) \simeq N^{\text{cyc}}_{\text{Bök}}(S \cdot N^i \mathcal{M})$$

$\mathcal{T}$ = Torsion f.g. $R$-modules

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Idea

Use Waldhausen’s localization sequence in $K$-theory to construct a localization sequence in $THH$.

$$THH(k) \to THH(R) \to THH(R \mid F)$$

Recall

$$THH(k) \cong THH(\mathcal{T}) := N^{\text{cyc}}_{\text{Bök}}(S \bullet \mathcal{T}) \cong N^{\text{cyc}}_{\text{Bök}}(S \bullet N^{i} \mathcal{T})$$

$$THH(R) \cong THH(\mathcal{M}) := N^{\text{cyc}}_{\text{Bök}}(S \bullet \mathcal{M}) \cong N^{\text{cyc}}_{\text{Bök}}(S \bullet N^{i} \mathcal{M})$$

$\mathcal{T} = \text{Torsion f.g. } R\text{-modules}$

$\mathcal{M} = \text{All f.g. } R\text{-modules}$
Use Waldhausen’s localization sequence in $K$-theory to construct a localization sequence in $THH$.

\[ THH(k) \to THH(R) \to THH(R \mid F) \]

Recall

\[ THH(k) = THH(\mathcal{T}) = N_{\text{Bök}}^{\text{cyc}}(S \cdot \mathcal{T}) \cong N_{\text{Bök}}^{\text{cyc}}(S \cdot N^i \mathcal{T}) \]
\[ THH(R) = THH(\mathcal{M}) = N_{\text{Bök}}^{\text{cyc}}(S \cdot \mathcal{M}) \cong N_{\text{Bök}}^{\text{cyc}}(S \cdot N^i \mathcal{M}) \cong N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w \mathcal{C}) \]

$\mathcal{T} =$ Torsion f.g. $R$-modules
$\mathcal{M} =$ All f.g. $R$-modules
$\mathcal{C} =$ Complexes of f.g. $R$-modules
$w =$ weak equivalences $=$ quasi-isomorphisms
Idea

Use Waldhausen’s localization sequence in $K$-theory to construct a localization sequence in $THH$.

$$THH(k) \to THH(R) \to THH(R | F)$$

Recall

$$THH(k) = THH(\mathcal{T}) = N_{\text{Bök}}^{\text{cyc}}(S \cdot \mathcal{T}) \simeq N_{\text{Bök}}^{\text{cyc}}(S \cdot N_{i}^{\text{t}} \mathcal{T}) \simeq N_{\text{Bök}}^{\text{cyc}}(S \cdot N_{w}^{\text{C}})$$

$$THH(R) = THH(\mathcal{M}) = N_{\text{Bök}}^{\text{cyc}}(S \cdot \mathcal{M}) \simeq N_{\text{Bök}}^{\text{cyc}}(S \cdot N_{i}^{\text{q}} \mathcal{M}) \simeq N_{\text{Bök}}^{\text{cyc}}(S \cdot N_{w}^{\text{C}})$$

$\mathcal{T} = \text{Torsion f.g. } R\text{-modules}$

$\mathcal{M} = \text{All f.g. } R\text{-modules}$

$\mathcal{C} = \text{Complexes of f.g. } R\text{-modules}$

$w = \text{weak equivalences = quasi-isomorphisms}$

$q = \text{mod torsion equivalences}$
Construction

\[ \text{THH}(k) \rightarrow \text{THH}(R) \rightarrow \text{THH}(R \mid F) \]

\[ N_{\text{Bök}}^{\text{cyc}}(\mathcal{S} \cdot N^w C^q) \]

Definition

\[ \text{THH}(R \mid F) = N_{\text{Bök}}^{\text{cyc}}(\mathcal{S} \cdot N^q C) \]
Construction

\[ \text{Definition} \]

\[ THH(R \mid F) = N^\text{cyc}_{\text{Bök}}(S \cdot N^q C) \]
Construction

\[ \text{THH}(k) \xrightarrow{\sim} \text{THH}(R) \xrightarrow{\sim} \text{THH}(R \mid F) \]

\[ \text{N}_{\text{cyc}}^{\text{Bök}}(S \cdot N^w C^q) \xrightarrow{\sim} \text{N}_{\text{cyc}}^{\text{Bök}}(S \cdot N^w C) \xrightarrow{\sim} \text{N}_{\text{cyc}}^{\text{Bök}}(S \cdot N^q C) \]

**Theorem (Waldhausen Square)**

*The square*

\[ (S \cdot N^w C^q) \xrightarrow{} (S \cdot N^q C^q) \]

\[ (S \cdot N^w C) \xrightarrow{} (S \cdot N^q C) \]
Construction

\[ \text{Theorem (Waldhausen/McCarthy)} \]

The square

\[
\begin{align*}
N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w_C \cdot C^q) &\rightarrow N_{\text{Bök}}^{\text{cyc}}(S \cdot N^q_C \cdot C^q) \\
&\sim \not\simeq \\
N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w_C) &\rightarrow N_{\text{Bök}}^{\text{cyc}}(S \cdot N^q_C)
\end{align*}
\]

is homotopy cartesian
Ausoni–Rognes Computations

Ausoni and Ausoni–Rognes compute

- $K(ku)$ and $K(\ell_p^\wedge)$
- $K(KU)$ and $K(L_p^\wedge)$

mod $p$ and $v_1$ assuming:

- Localization sequence in $K$-theory

\[
K(H\mathbb{Z}) \to K(ku) \to K(KU)
\]
\[
K(H\mathbb{Z}_p^\wedge) \to K(\ell_p^\wedge) \to K(L_p^\wedge)
\]

- Localization sequence in $THH$

\[
THH(H\mathbb{Z}) \to THH(ku) \to THH(KU)
\]
\[
THH(H\mathbb{Z}_p^\wedge) \to THH(\ell_p^\wedge) \to THH(L_p^\wedge)
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mod $p$ and $v_1$ assuming:

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$$K(H\mathbb{Z}) \to K(ku) \to K(KU)$$
$$K(H\mathbb{Z}_p^\wedge) \to K(\ell_p^\wedge) \to K(L_p^\wedge)$$

- Localization sequence in $THH$

$$THH(H\mathbb{Z}) \to THH(ku) \to THH(KU)$$
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$$K(H\mathbb{Z}) \to K(ku) \to K(KU)$$
$$K(H\mathbb{Z}_p^\wedge) \to K(\ell_p^\wedge) \to K(L_p^\wedge)$$

- Localization sequence in $THH$

$$THH(H\mathbb{Z}) \to THH(ku) \to THH(KU)$$
$$THH(H\mathbb{Z}_p^\wedge) \to THH(\ell_p^\wedge) \to THH(L_p^\wedge)$$
The localization sequence for $K(KU)$

Use Waldhausen’s square again:

**Theorem (Waldhausen)**

The square

\[
\begin{array}{ccc}
\text{Ob}(S \cdot N^w C^q) & \rightarrow & \text{Ob}(S \cdot N^q C^q) \\
\downarrow & & \downarrow \\
\text{Ob}(S \cdot N^w C) & \rightarrow & \text{Ob}(S \cdot N^q C)
\end{array}
\]

is homotopy cartesian

But now:

$C = \text{finite cell } ku\text{-modules}$

$w = \text{weak equivalences}$

$q = \text{maps that become weak equivalences after inverting Bott element}$

Devissage theorem identifying $\text{Ob}(S \cdot N^w C^q)$ as $K(H\mathbb{Z})$
The localization sequence for $K(KU)$

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The square

$$
\begin{align*}
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\downarrow & \downarrow \\
\text{Ob}(S \cdot N^w C) & \to \text{Ob}(S \cdot N^q C)
\end{align*}
$$

is homotopy cartesian

But now:

- $C =$ finite cell $ku$-modules
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The localization sequence for \( K(KU) \)

Use Waldhausen’s square again:

**Theorem (Waldhausen)**

The square

\[
\begin{array}{c}
\text{Ob}(S \cdot N^w C^q) \\
\downarrow \\
\text{Ob}(S \cdot N^w C)
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{Ob}(S \cdot N^q C^q) \\
\downarrow \\
\text{Ob}(S \cdot N^q C)
\end{array}
\]

is homotopy cartesian

But now:
\( \mathcal{C} = \text{finite cell } ku\text{-modules} \)
\( w = \text{weak equivalences} \)
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Devissage theorem identifying \( \text{Ob}(S \cdot N^w C^q) \) as \( K(H\mathbb{Z}) \)
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is homotopy cartesian

But now:
- $\mathcal{C} = \text{finite cell } ku\text{-modules}$
- $w = \text{weak equivalences}$
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Devissage theorem identifying $\text{Ob}(S \cdot N^w C^q)$ as $K(H\mathbb{Z})$
Categories $N^w_m C$ not exact categories, but are *spectral categories*

Use natural mapping spectra in $C$.

Get mapping spectra for diagram categories

\[ N^w_m C, \quad S_n N^w_m C, \quad S_n N^w_m C^q, \quad \text{etc.} \]

Then apply $N^\text{cyc} Bök$ to square

\[
\begin{align*}
S \cdot N^w C^q &\to S \cdot N^q C^q \\
\downarrow &\downarrow \\
S \cdot N^w C &\to S \cdot N^q C
\end{align*}
\]
Categories $N_m^wC$ not exact categories, but are *spectral categories*

Use natural mapping spectra in $C$.

Get mapping spectra for diagram categories

\[ N_m^wC, \quad S_nN_m^wC, \quad S_nN_m^wC^q, \quad \text{etc.} \]

Then apply $N_{\text{Bök}}^{\text{cyc}}$ to square

\[
\begin{align*}
S \cdot N_m^wC^q & \rightarrow S \cdot N_q^wC^q \\
\downarrow & \\
S \cdot N_m^wC & \rightarrow S \cdot N_q^wC
\end{align*}
\]
Categories $N^w_m \mathcal{C}$ not exact categories, but are *spectral categories*

Use natural mapping spectra in $\mathcal{C}$.

Get mapping spectra for diagram categories

$$N^w_m \mathcal{C}, \quad S_n N^w_m \mathcal{C}, \quad S_n N^w_m \mathcal{C}^q, \quad \text{etc.}$$

Then apply $N^{cyc}_{\text{Bök}}$ to square

$$S_\bullet N^\bullet w \mathcal{C}^q \rightarrow S_\bullet N^\bullet q \mathcal{C}^q \quad \downarrow \quad \downarrow$$

$$S_\bullet N^\bullet w \mathcal{C} \rightarrow S_\bullet N^\bullet q \mathcal{C}$$
First localization sequence for \textit{THH}

Categories $N^w_mC$ not exact categories, but are \textit{spectral categories}

Use natural mapping spectra in $\mathcal{C}$.

Get mapping spectra for diagram categories

$$N^w_mC, \quad S_nN^w_mC, \quad S_nN^w_C^q, \quad \text{etc.}$$

Then apply $N^{cyc}_{\text{B"ok}}$ to square

$$S\cdot N^w_C^q \rightarrow S\cdot N^q_C^q \downarrow \downarrow$$

$$S\cdot N^w_C \rightarrow S\cdot N^q_C$$
First localization sequence for \(THH\)

Categories \(N'_mC\) not exact categories, but are \emph{spectral categories}

Use natural mapping spectra in \(C\).

Get mapping spectra for diagram categories

\[
N'_mC, \quad S_nN'_mC, \quad S_nN'_mC^q, \quad \text{etc.}
\]

Then apply \(N^\text{cyc}_{Bök}\) to square

\[
\begin{array}{ccc}
S\cdot N'_mC^q & \rightarrow & S\cdot N^qC^q \\
\downarrow & & \downarrow \\
S\cdot N'_mC & \rightarrow & S\cdot N^qC
\end{array}
\]
First localization sequence for $THH$

Categories $N_m^w C$ not exact categories, but are *spectral categories*

Use natural mapping spectra in $C$.

Get mapping spectra for diagram categories

$$N_m^w C, \quad S_n N_m^w C, \quad S_n N_m^w C^q, \quad \text{etc.}$$

Then apply $N_{Bök}^{cyc}$ to square

\[
\begin{align*}
S \cdot N_m^w C^q &\rightarrow S \cdot N_q^q C^q \\
\downarrow &\downarrow \\
S \cdot N_m^w C &\rightarrow S \cdot N_q^q C
\end{align*}
\]
Categories $N^w_mC$ not exact categories, but are *spectral categories*

Use natural mapping spectra in $C$.

Get mapping spectra for diagram categories

\[ N^w_mC, \quad S_nN^w_mC, \quad S_nN^w_mC^q, \quad \text{etc.} \]

Then apply $N^{cyc}_{\text{B"ok}}$ to square

\[
\begin{array}{ccc}
\bullet & N^w_mC^q & \rightarrow & S \cdot N^qC^q \\
\downarrow & & & \downarrow \\
\bullet & S \cdot N^wC & \rightarrow & \bullet \cdot N^qC
\end{array}
\]
First localization sequence for \( THH \) II

Get homotopy (co)cartesian square

\[
\begin{array}{ccc}
N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C^q) & \rightarrow & N_{\text{Bök}}^{\text{cyc}}(S \cdot N^q C^q) \\
\downarrow & & \downarrow \\
N_{\text{Bök}}^{\text{cyc}}(S \cdot N^w C) & \rightarrow & N_{\text{Bök}}^{\text{cyc}}(S \cdot N^q C)
\end{array}
\]

and cofiber sequence

\[
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But \( THH(N^q C) \simeq THH(KU) \)
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First localization sequence for $THH$ II

Get homotopy (co)cartesian square

\[
\begin{array}{c}
\mathcal{N}_{Bök}^{\text{cyc}}(S \cdot N^w C^q) \to \mathcal{N}_{Bök}^{\text{cyc}}(S \cdot N^q C^q) \\
\downarrow \quad \downarrow \\
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\end{array}
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\downarrow & \downarrow \\
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Quillen’s cofiber sequence

\[ K(k) \to K(R) \to K(F) \]

generalizes to Thomason–Trobaugh’s cofiber sequence

\[ K^B(X \text{ on } Y) \to K^B(X) \to K^B(X \setminus Y) \]

“on” means “supported on”

Using \( \text{THH} \) of spectral categories of perfect complexes, get a cofiber sequence

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\[ X \text{ on } (X - Y) \]
Localization sequence for $THH$ of schemes

For open $U \supset Y$, $K(X \text{ on } Y) \simeq K(U \text{ on } Y)$

$$K(X - Y) \simeq K(X \text{ on } (X - Y))$$

When $X = \text{Spec } R$ for any commutative ring $R$, we can use $C = C_{HR}$ finite cell $HR$-modules.

Then cofiber sequence

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Why Thomason–Trobaugh sequence different from the Hesselholt-Madsen sequence?

Hesselholt–Madsen Sequence
- Treat category of complexes as an exact category
- Mapping spectra always Eilenberg–Mac Lane spectrum – no homotopy groups except in degree zero
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Simplicially enriched Waldhausen categories

Waldhausen Category

- Notion of weak equivalence
- Notion of cofibration
- Pushouts over cofibration (including sums)
- Nice relationship between weak equivalence and cofibration

Now also assume simplicial enrichment for mapping spaces

Without loss of generality: Modern homotopy theory says that this structure always arises and plays nicely with cofibrations and weak equivalences.
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Connective spectral enrichment

From mapping space $\mathcal{C}(X, Y)$ get connective spectrum from gamma space

$$\mathcal{C}(X, Y)_{\Gamma_m} = \mathcal{C}(X, \bigvee_{\gamma} Y) \simeq \prod_{m} \mathcal{C}(X, Y)$$

Use this spectral enrichment to construct a new $THH$.

$$W^\Gamma_{THH}(C) := N_{\text{Bök}}^{\text{cyC}}(S \cdot \mathcal{C}^\Gamma)$$
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Trace Map

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\[ W^\Gamma \text{THH}(C) := N^{\text{cyc}}_{\text{Bök}}(S \cdot C^\Gamma) \]

Not hard to see \( N^{\text{cyc}}_{\text{Bök}}(S \cdot C^\Gamma) \cong N^{\text{cyc}}_{\text{Bök}}(S \cdot N^w C^\Gamma) \)

Get trace map

\[ K(C) = \text{Ob}(S \cdot N^w C) \to N^{\text{cyc}}_{\text{Bök}}(S \cdot N^w C^\Gamma) \cong W^\Gamma \text{THH}(C) \]

as inclusion of objects

When \( C \) has intrinsic mapping spectra, trace map factors through this non-connective enrichment.
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Connective and non-connective enrichments

Connective enrichment vs. non-connective enrichment

\[ C(X, Y)^\Gamma (n) = |C(X, \bigvee_{S^n} Y)| \quad C(X, Y)^S (n) = |C(X, Y \otimes S^n)| \]

Canonical map

\[ C(X, Y)^\Gamma \to C(X, Y)^S \]

often connective cover,
e.g., when \( C(X, Y) \to C(\Sigma X, \Sigma Y) \) is a weak equivalence

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M.A. Mandell (IU)  Localization Sequences in THH  Jan 2012
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Do we actually get something different?

Let $\mathcal{E}$ be an exact category, viewed as a Waldhausen category with weak equivalences the isomorphisms and mapping spaces discrete.

Then $W^\Gamma THH(\mathcal{E})$ is the Dundas–McCarthy $THH(\mathcal{E})$.

Now let $\mathcal{E}$ be the exact category of locally free sheaves on a quasi-projective variety $X$.

Can give $\mathcal{E}$ a non-connective spectra enrichment $\mathcal{E}^S$ that correctly captures the fact that Ext in $\mathcal{E}$ can be non-trivial.

When $X$ is affine $\mathcal{E} \simeq \mathcal{E}^S$ and $THH(\mathcal{E}) \simeq THH(\mathcal{E}^S)$.

Using $\mathcal{E}^S$, $\pi_* THH(\mathcal{E}^S)$ is a quasi-coherent sheaf over $X$. This does not hold in general for $\pi_* THH(\mathcal{E})$. 
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Let $\mathcal{C}_R$ be the Waldhausen category of finite cell $R$-modules for $R$ a connective EKMM $S$-algebra or $R$ a simplicial $R$-algebra. Then $W^\Gamma THH(\mathcal{C}_R) \simeq THH(R)$.

Does not hold if we do not assume connective.

Theorem generalizes to any simplicially enriched Waldhausen category $\mathcal{C}$ that has a set $Q$ of generators such that the mapping spectra between objects in $Q$ are connective.
What about for categories of \( R \)-modules in spectra

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**Theorem**

Let $R$ be a connective EKMM $S$-algebra with $\pi_0 R$ connective and let $\mathcal{P}$ be the category of cell $R$-algebras that have finitely many non-zero homotopy groups all of which are finitely generated. Then $W^\Gamma \text{THH}(\mathcal{P}) \simeq \text{THH}(\pi_0 R)$.

In particular $W^\Gamma \text{THH}(\mathcal{P})$ has zero negative homotopy groups. Using the natural (non-connective) mapping spectra, usually get negative homotopy groups for $\text{THH}(\mathcal{P})$ unless $R \simeq H\pi_0 R$.

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A devissage theorem

**Theorem**

Let $R$ be a connective EKMM $S$-algebra with $\pi_0 R$ connective and let $\mathcal{P}$ be the category of cell $R$-algebras that have finitely many non-zero homotopy groups all of which are finitely generated. Then $W^\Gamma THH(\mathcal{P}) \simeq THH(\pi_0 R)$.

In particular $W^\Gamma THH(\mathcal{P})$ has zero negative homotopy groups. Using the natural (non-connective) mapping spectra, usually get negative homotopy groups for $THH(\mathcal{P})$ unless $R \simeq H\pi_0 R$.

For $R = ku$, $\mathcal{P} \simeq C^q$
Corollary: Localization sequence for the $THH$ of topological $K$-theory

Define $W^\Gamma THH(ku \mid KU)$ as $N^{cyc}_{\text{Bök}}(S \cdot N^q C^\Gamma)$

Corollary

The cofiber sequence

$$N^{cyc}_{\text{Bök}}(S \cdot N^w (C^\Gamma)^q) \rightarrow N^{cyc}_{\text{Bök}}(S \cdot N^w C^\Gamma) \rightarrow N^{cyc}_{\text{Bök}}(S \cdot N^q C^\Gamma)$$

is weakly equivalent to a cofiber sequence

$$THH(\mathbb{Z}) \rightarrow THH(ku) \rightarrow W^\Gamma THH(ku \mid KU)$$

where the map $THH(\mathbb{Z}) \rightarrow THH(ku)$ is a certain previously known transfer map.
For $\mathcal{C}$ the category of cell $ku$-modules

Connective and non-connective spectral enrichment give two different localization sequences:

\[
\begin{align*}
K(H\mathbb{Z}) & \to K(ku) \to K(KU) \\
\downarrow & \downarrow & \downarrow \\
THH(H\mathbb{Z}) & \to THH(ku) \to W[THH(ku \mid KU)]
\end{align*}
\]

THH($ku$ on $H\mathbb{Z}$)  THH($ku$)  THH($KU$)
For $\mathcal{C}$ the category of cell $ku$-modules

Connective and non-connective spectral enrichment give two different localization sequences: