Localization Sequences in *THH*

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Topology Seminar
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Overview

Localization Sequences in *THH*

- Joint work with Andrew Blumberg

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Localization Sequences in *THH* and *TC*

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Localization Sequences in *THH* and *TC* II

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- Preprint: arXiv:1111.4003 [math.KT]

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2nd Goal: Make sense of the *THH* of Waldhausen categories

3rd Goal: Understand the relationship to already known localization sequences in *THH* and *TC*

Quillen Localization Sequence

Let R be a discrete valuation ring, with residue field k and field of fractions F. Then there is a cofibration sequence of K-theory spectra

$$K(k) \to K(R) \to K(F).$$

$\underline{k = R/\pi}$ $\quad F = R[\pi^{-1}]$

DVR = local ring
$\underline{\text{PID}}$

$R = \hat{\mathbb{Z}}_p$

PID w/ unique irreducible (up to unit)

$k = \mathbb{Z}/p$

π

$F = \hat{\mathbb{Q}}_p$

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This uses the K-theory of abelian categories.

Secretly $K(k)$ is really the K-theory of the category of finitely generated torsion R-modules.

(Devissage theorem.)

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Hesselholt–Madsen Theorem

Let R be a complete discrete valuation ring, with residue field k perfect of characteristic $\boxed{p > 2}$ and field of fractions F of characteristic zero, containing the p^n-th roots of unity. Then:

- $K(F; \mathbb{Z}/p^n)$ can be computed in terms of the De Rham–Witt complex.
- F satisfies the Lichtenbaum-Quillen conjecture. (p=2 by Voevodsky)

Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.

$R = \varprojlim R/\pi^n$

Argument

1. McCarthy Theorem: If $A \to B$ is nilpotent with surjective kernel
$$\begin{array}{ccc} K(A)^\wedge_p & \longrightarrow & K(B)^\wedge_p \\ \downarrow & & \downarrow \\ TC(A) & \longrightarrow & TC(B) \end{array}$$
is homotopy cartesian.
2. Quillen–Krasner Theorem: If k is a perfect field of characteristic p, then $K(k)^\wedge_p \simeq H\mathbb{Z}^\wedge_p$.
3. Suslin K-theory continuity results imply $K(R)^\wedge_p \simeq \operatorname{holim} K(R/\pi^n)^\wedge_p$.

Hesselholt–Madsen then show:
$$K(k)^\wedge_p \simeq TC(k)[0,\infty) \qquad K(R)^\wedge_p \simeq TC(R)[0,\infty)$$

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$1 = k$

$A = R/(\pi^n)$

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   (very specialized!)

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Key step to enable computation

Identify the cofiber of $TC(k) \to TC(R)$ in intrinsic terms.

Identify the cofiber of $THH(k) \to THH(R)$ in intrinsic terms.

Note: Cofiber is not $THH(F)$

Example

$R = \mathbb{Z}_p^\wedge, \qquad k = \mathbb{Z}/p, \qquad F = \mathbb{Q}_p^\wedge$

- $THH(\mathbb{Z}/p) = \bigvee \Sigma^{2n} H\mathbb{Z}/p$
- $THH(\mathbb{Z}_p^\wedge) = \bigvee \Sigma^{2n-1} H\mathbb{Z}_p^\wedge/n$
- $THH(\mathbb{Q}_p^\wedge) = H\mathbb{Q}_p^\wedge$

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Idea

Use Waldhausen's localization sequence in K-theory to construct a localization sequence in THH.

$$THH(k) \to THH(R) \to THH(R \mid F)$$

Recall

$$THH(k) = THH(\mathcal{T}) = N^{cyc}_{\text{Bök}}(S_\bullet \mathcal{T}) \simeq N^{cyc}_{\text{Bök}}(S_\bullet N^i_\bullet \mathcal{T})$$
$$THH(R) = THH(\mathcal{M}) = N^{cyc}_{\text{Bök}}(S_\bullet \mathcal{M}) \simeq N^{cyc}_{\text{Bök}}(S_\bullet N^i_\bullet \mathcal{M})$$

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$X \overset{\simeq}{\Rightarrow} Y \overset{\simeq}{\Rightarrow} Z$

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$\mathcal{T}$ = Torsion f.g. R-modules
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$\mathcal{C}$ = Complexes of f.g. R-modules
w = weak equivalences = quasi-isomorphisms

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q-acyclic objects of $\mathcal{C}$

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w = weak equivalences = quasi-isomorphisms
q = mod torsion equivalences

Construction

$$\begin{array}{ccccc} THH(k) & \longrightarrow & THH(R) & \longrightarrow & THH(R \mid F) \\ \| & & \| & & \\ N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^q) & & N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}) & & N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C}) \end{array}$$

Definition

$THH(R \mid F) = N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C})$

Construction

$$\begin{tikzcd}
THH(k) \arrow[r] \arrow[d, equal] & THH(R) \arrow[r] \arrow[d, equal] & THH(R \mid F) \arrow[d, equal] \\
N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^q) \arrow[r] \arrow[rr, bend right] & N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}) \arrow[r] & N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C})
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Theorem (Waldhausen Square)

The square

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(S_\bullet N^w_\bullet \mathcal{C}^q) \arrow[r] \arrow[d] & (S_\bullet N^q_\bullet \mathcal{C}^q) \arrow[d] \\
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Dundas

Theorem (Waldhausen/McCarthy)

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$$
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N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^q) & \longrightarrow & N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C}^q) \simeq \cancel{k} \\
\downarrow & & \downarrow \\
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$$

is homotopy cartesian

Ausoni–Rognes Computations

Ausoni and Ausoni–Rognes compute

- $K(ku)$ and $K(\ell_p^\wedge)$
- $K(KU)$ and $K(L_p^\wedge)$

mod p and v_1 assuming:

- Localization sequence in K-theory

$$K(H\mathbb{Z}) \to K(ku) \to K(KU)$$
$$K(H\mathbb{Z}_p^\wedge) \to K(\ell_p^\wedge) \to K(L_p^\wedge)$$

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B-M

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current preprint.

The localization sequence for $K(KU)$

Use Waldhausen's square again:

Theorem (Waldhausen)

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$$\begin{array}{ccc} \mathrm{Ob}(S_\bullet N^w_\bullet \mathcal{C}^q) & \longrightarrow & \mathrm{Ob}(S_\bullet N^q_\bullet \mathcal{C}^q) \\ \downarrow & & \downarrow \\ \mathrm{Ob}(S_\bullet N^w_\bullet \mathcal{C}) & \longrightarrow & \mathrm{Ob}(S_\bullet N^q_\bullet \mathcal{C}) \end{array}$$

is homotopy cartesian

But now:
$\mathcal{C}$ = finite cell *ku*-modules
w = weak equivalences
q = maps that become weak equivalences after inverting Bott element

Devissage theorem identifying $\mathrm{Ob}(S_\bullet N^w_\bullet \mathcal{C}^q)$ as $K(H\mathbb{Z})$

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First localization sequence for *THH*

Categories $\underbrace{N^w_m\mathcal{C}}$ not exact categories,

but are *spectral categories*

Use natural mapping spectra in $\mathcal{C}$.

Get mapping spectra for diagram categories

$$N^w_m\mathcal{C}, \qquad S_n N^w_m\mathcal{C}, \qquad S_n N^w_m\mathcal{C}^q, \qquad \text{etc.}$$

Then apply $N^{cyc}_{\text{Bök}}$ to square

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but are *spectral categories*

Use natural mapping spectra in $\mathcal{C}$.

Get mapping spectra for diagram categories

$$N^w_m\mathcal{C}, \qquad S_nN^w_m\mathcal{C}, \qquad S_nN^w_m\mathcal{C}^q, \qquad \text{etc.}$$

Then apply $N^{cyc}_{\text{Bök}}$ to square

$$\begin{tikzcd} S_\bullet N^w_\bullet\mathcal{C}^q \arrow[r] \arrow[d] & S_\bullet N^q_\bullet\mathcal{C}^q \arrow[d] \\ S_\bullet N^w_\bullet\mathcal{C} \arrow[r] & S_\bullet N^q_\bullet\mathcal{C} \end{tikzcd}$$

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First localization sequence for *THH* II

Get homotopy (co)cartesian square

$$\begin{array}{ccc} N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^q) & \longrightarrow & N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C}^q) \\ \downarrow & & \downarrow \\ N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}) & \longrightarrow & N^{cyc}_{\text{Bök}}(S_\bullet N^q_\bullet \mathcal{C}) \end{array}$$

and cofiber sequence

$$THH(N^w_\bullet \mathcal{C}^q) \to THH(ku) \to THH(N^q_\bullet \mathcal{C})$$

But $THH(N^q_\bullet \mathcal{C}) \simeq THH(KU)$
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How does this localization sequence fit in?

Quillen's cofiber sequence

$$K(k) \to K(R) \to K(F)$$

generalizes to Thomason–Trobaugh's cofiber sequence

$$K^B(X \text{ on } Y) \to K^B(X) \to K^B(X - Y)$$

"on" means "supported on"

Using *THH* of spectral categories of perfect complexes, get a cofiber sequence

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Localization sequence for *THH* of schemes

For open $U \supset Y$, $K(X \text{ on } Y) \simeq K(U \text{ on } Y))$

$$K(X - Y) \simeq K(X \text{ on } (X - Y))$$

When $X = \operatorname{Spec} R$ for any commutative ring R,
we can use $\mathcal{C} = \mathcal{C}_{HR}$ finite cell HR-modules.

Then cofiber sequence

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Why Thomason–Trobaugh sequence different from the Hesselholt-Madsen sequence?

Hesselholt–Madsen Sequence

- Treat category of complexes as an *exact* category
- Mapping spectra always Eilenberg–Mac Lane spectrum – no homotopy groups except in degree zero
- Meaning of mapping spectra?

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Simplicially enriched Waldhausen categories

Waldhausen Category

- Notion of weak equivalence
- Notion of cofibration
- Pushouts over cofibration (including sums)
- Nice relationship between weak equivalence and cofibration

Now also assume simplicial enrichment for mapping *spaces*

Without loss of generality: Modern homotopy theory says that this structure always arises and plays nicely with cofibrations and weak equivalences.

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Connective spectral enrichment

From mapping space $\mathcal{C}(X, Y)$ get connective spectrum from gamma space

$$\mathcal{C}(X, Y)^{\Gamma}_{m} = \mathcal{C}(X, \bigvee_{m} Y) \simeq \prod_{m} \mathcal{C}(X, Y)$$

$$\mathcal{C}(\Sigma ku, ku) \simeq \Omega ku$$

Use this spectral enrichment to construct a new *THH*.

$$\mathcal{C}(\Sigma ku, ku)^{\Gamma} \simeq (\Omega ku)[0,\infty)$$

Definition

$$W^{\Gamma} THH(\mathcal{C}) := N^{cyc}_{\text{Bök}}(S_{\bullet}\mathcal{C}^{\Gamma})$$

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Get trace map

$$K(\mathcal{C}) = \mathrm{Ob}(S_\bullet N^w_\bullet \mathcal{C}) \to N^{cyc}_{\text{Bök}}(S_\bullet N^w_\bullet \mathcal{C}^\Gamma) \simeq W^\Gamma THH(\mathcal{C})$$

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When $\mathcal{C}$ has intrinsic mapping spectra, trace map factors through this non-connective enrichment.

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Connective and non-connective enrichments

Connective enrichment vs. non-connective enrichment

$$\mathcal{C}(X,Y)^{\Gamma}(n) = |\mathcal{C}(X, \bigvee_{S^n_\bullet} Y)| \qquad \mathcal{C}(X,Y)^{S}(n) = |\mathcal{C}(X, Y \otimes S^n)|$$

Canonical map

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Do we actually get something different?

Let $\mathcal{E}$ be an exact category, viewed as a Waldhausen category with weak equivalences the isomorphisms and mapping spaces discrete.

Then $W^{\Gamma}\mathit{THH}(\mathcal{E})$ is the Dundas–McCarthy $\mathit{THH}(\mathcal{E})$.

Now let $\mathcal{E}$ be the exact category of locally free sheaves on a quasi-projective variety X.

Can give $\mathcal{E}$ a non-connective spectra enrichment $\mathcal{E}^S$ that correctly captures the fact that Ext in $\mathcal{E}$ can be non-trivial.

When X is affine $\mathcal{E} \simeq \mathcal{E}^S$ and $\mathit{THH}(\mathcal{E}) \simeq \mathit{THH}(\mathcal{E}^S)$.

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Can give $\mathcal{E}$ a non-connective spectra enrichment $\mathcal{E}^S$ that correctly captures the fact that Ext in $\mathcal{E}$ can be non-trivial.

When X is affine $\mathcal{E} \simeq \mathcal{E}^S$ and $THH(\mathcal{E}) \simeq THH(\mathcal{E}^S)$.

Using $\mathcal{E}^S$, $\pi_* THH(\mathcal{E}^S)$ is a quasi-coherent sheaf over X. This does not hold in general for $\pi_* THH(\mathcal{E})$.

What about for categories of R-modules in spectra

Theorem (Sphere Theorem)

Let $\mathcal{C}_R$ be the Waldhausen category of finite cell R-modules for R a connective EKMM S-algebra or R a simplicial R-algebra. Then $W^\Gamma THH(\mathcal{C}_R) \simeq THH(R)$.

Does not hold if we do not assume connective.

Theorem generalizes to any simplicially enriched Waldhausen category $\mathcal{C}$ that has a set Q of generators such that the mapping spectra between objects in Q are connective.

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Let R be a connective EKMM S-algebra with $\pi_0 R$ connective and let $\mathcal{P}$ be the category of cell R-algebras that have finitely many non-zero homotopy groups all of which are finitely generated. Then $W^\Gamma THH(\mathcal{P}) \simeq THH(\pi_0 R)$.

In particular $W^\Gamma THH(\mathcal{P})$ has zero negative homotopy groups. Using the natural (non-connective) mapping spectra, usually get negative homotopy groups for $THH(\mathcal{P})$ unless $R \simeq H\pi_0 R$.

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Corollary: Localization sequence for the *THH* of topological *K*-theory

Define $W^{\Gamma} THH(ku \mid KU)$ as $N^{cyc}_{\text{Bök}}(S_{\bullet} N^{q}_{\bullet} \mathcal{C}^{\Gamma})$

Corollary

The cofiber sequence

$$N^{cyc}_{\text{Bök}}(S_{\bullet} N^{w}_{\bullet} (\mathcal{C}^{\Gamma})^{q}) \to N^{cyc}_{\text{Bök}}(S_{\bullet} N^{w}_{\bullet} \mathcal{C}^{\Gamma}) \to N^{cyc}_{\text{Bök}}(S_{\bullet} N^{q}_{\bullet} \mathcal{C}^{\Gamma})$$

is weakly equivalent to a cofiber sequence

$$THH(\mathbb{Z}) \to THH(ku) \to W^{\Gamma} THH(ku \mid KU)$$

where the map $THH(\mathbb{Z}) \to THH(ku)$ is a certain previously known transfer map.

Summary

For $\mathcal{C}$ the category of cell *ku*-modules

Connective and non-connective spectral enrichment give two different localization sequences:

$$\begin{tikzcd}
K(H\mathbb{Z}) \arrow[r] \arrow[d] & K(ku) \arrow[r] \arrow[d] & K(KU) \arrow[d] \\
THH(H\mathbb{Z}) \arrow[r] & THH(ku) \arrow[r] & W^{\Gamma} THH(ku \mid KU)
\end{tikzcd}$$

$$THH(ku \text{ on } H\mathbb{Z}) \qquad THH(ku) \qquad THH(KU)$$

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uses conn mapping spectra

uses non-con mapping spectra

$THH(N?e^{\Gamma})$