Large Scale Open Subsets of Configuration Spaces and the Foundations of Factorization Homology

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Indiana University

Topology Seminar

February 22, 2021

Overview

Joint work with Andrew Blumberg

Goal: Construct norms for positive codimensional (closed) subgroups of compact Lie groups

Intuition is $N_H^G A \simeq \int_{G/H} A$

- Looks good for G/H finite index: $N_H^G A \simeq A^{(G/H)}$
- Looks good for $G = \mathbb{T} = S^1$, $H = e$: $N_e^{\mathbb{T}} A \simeq THH(A)$

Project Stages

- Foundations of factorization homology (nonequivariantly)
- Equivariant homotopy of smash power
- Equivariant homotopy of equivariant factorization homology

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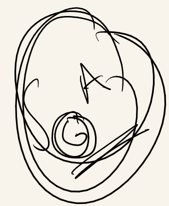

Foundations of factorization homology

Need arguments that

- Avoid simplicial approximation (quasicategorical methods)
- Avoid local-to-global methods starting from a point or configuration

Construct covers of configuration spaces by "large scale" open subsets with intersection combinatorics and homotopy types mirroring a "Quillen Theorem A" comparison space

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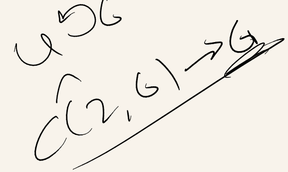

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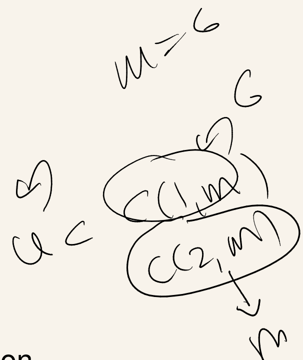

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$(\infty, 1)$-categories

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$(\infty, 1)$-categories

- Topological categories / continuous functors (e.g., cofibrant fibrant orthogonal spectra)
- Relative topological categories / continuous functors that preserve weak equivalences (e.g., all orthogonal spectra with usual weak equivalences)

$$\int_M A \to \int_N A$$

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Manifold category

- Objects: Smooth d-manifolds with finitely many components
- Morphisms: Smooth embeddings
- Notation: $\mathscr{E}(M, N)$

Framings (!)

Nonunital Variant

Only allow maps that are surjective on π_0 ($\mathscr{E}(\emptyset, M)$ empty rather than 1-point)

Factorization homology

For a fixed "disk algebra" A, $\int_M A$ is a continuous functor of M from $\mathscr{E}$ to orthogonal spectra

$\int_M A$ is a relative continuous functor of (M, A) from $\mathscr{E} \times \mathscr{A}lg^{\flat}_{\mathcal{D}}$ to orthogonal spectra

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- Objects $\mathbb{N} = \{0, 1, 2, \ldots\}$
- Morphisms $\mathscr{D}(m, n) = \mathscr{E}(\mathbb{R}^d \times \{1, \ldots, m\}, \mathbb{R}^d \times \{1, \ldots, n\})$

Little disk category

Objects $\mathbb{N}$, maps from framed little disk operad D.

A morphism in $\mathcal{D}(m, n)$ consists of:

- Affine linear maps $D^d \to D^d$ that are orthogonal up to scaling and
- Images of open unit disks are disjoint

Inclusion of $\mathcal{D}$ in $\mathscr{D}$ is an equivalence

Disk algebra

A strict symmetric monoidal functor from $\mathcal{D}$ or $\mathscr{D}$ to orthogonal spectra

Factorization homology

$\int_M A$ is the (homotopy) left Kan extension of A along the inclusion of $\mathcal{D}$ (or $\mathscr{D}$) in $\mathscr{E}$

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m n

$\mathscr{D}(0, n) = \emptyset$

$\mathscr{E}$

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$B^d \cong \mathbb{R}^d$

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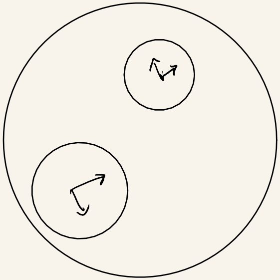

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Properties of factorization homology (and the norm)

Factorization homology

1. Symmetric monoidal in the manifold: $\int_{M\amalg N} A \simeq \int_M A \wedge \int_N A$
2. Symmetric monoidal in the algebra: $\int_M (A \wedge B) \simeq \int_M A \wedge \int_M B$
3. Bundle property: "$\int_B (\int_{F\times\mathbb{R}^c} A)$"$\simeq \int_E A$
4. Commutative algebra property: If A is a commutative algebra, $\int_M A \simeq A \otimes M$
5. Gluing property: "$(\int_M A) \wedge_{(\int_{L\times\mathbb{R}} A)} (\int_N A)$"$\simeq \int_{M\cup_L N} A$

The norm

1. If $H < G$ is finite index and $K < H$, then $N_K^G A \simeq (N_K^H A)^{(G/H)}$
2. $N_H^G(A \wedge B) \simeq N_H^G A \wedge N_H^G B$
3. For $K < H < G$, $N_H^G N_K^H \simeq N_K^G A$
4. Restricted to commutative H-algebras, N_H^G is the free functor to commutative G-algebras
5. (No analogue)

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Properties of factorization homology (and the norm)

Factorization homology

1. Symmetric monoidal in the manifold: $\int_{M \amalg N} A \simeq \int_M A \wedge \int_N A$
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1. If $H < G$ is finite index and $K < H$, then $N_K^G A \simeq (N_K^H A)^{(G/H)}$
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Construction of factorization homology

Factorization homology $\int_{(-)} A$ is (homotopy) left Kan extension of A along $\mathcal{D} \to \mathscr{E}$

Notation: Write $\mathcal{E}_M(m)$ for $\mathscr{E}(\mathbb{R}^d \times \{1, \ldots, m\}, M)$ (contravariant functor of $\mathscr{D}$ or $\mathcal{D}$)

Homotopy coend construction of homotopy left Kan extension

$$\int_M A = \mathrm{hocoend}_{\mathcal{D}}(\mathcal{E}_M(-)_+ \wedge A^{(-)})$$

Bar construction for homotopy coend $B(\mathcal{E}_M, \mathcal{D}, A)$

$$B_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \wedge A^{(k_0)} \qquad \mathcal{E}_M(k) \underset{k}{\bullet} A^{(k)}$$

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Smaller bar constructions

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Note on monoidality in manifold variable

$$\mathcal{E}_M(m) \times \mathcal{E}_N(n) \to \mathcal{E}_{M \amalg N}(m+n) \quad \Longrightarrow \quad B(\mathcal{E}_M, \mathcal{D}, A) \wedge B(\mathcal{E}_N, \mathcal{D}, A) \to B(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

Monoidal but not symmetric

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Monoidal but not symmetric

Smaller bar constructions

Bar construction $\tilde{B}(\mathcal{E}_M, \mathcal{D}, A)$

$$\tilde{B}_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k) \; \overset{\circlearrowleft}{\underset{k}{\bullet}} \; A^{(k)}$$

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$\mathbb{R}^d \times \{1, \dots, k\}^n$ / Σ_k

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$$\tilde{B}_1 = \bigvee_{k_0,k_1} (\mathcal{E}_M(k_1) \underset{\Sigma_{k_1}}{\times} \mathcal{D}(k_0,k_1))_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_1) \ \overset{\circlearrowleft}{\underset{k_1}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_0}{\bullet}} \ A^{(k_0)}$$

$$\tilde{B}_2 = \bigvee_{k_0,k_1,k_2} (\mathcal{E}_M(k_2) \underset{\Sigma_{k_2}}{\times} \mathcal{D}(k_1,k_2) \underset{\Sigma_{k_1}}{\times} \mathcal{D}(k_0,k_1))_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_2) \ \overset{\circlearrowleft}{\underset{k_2}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_1}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_0}{\bullet}} \ A^{(k_0)}$$

etc

$$\tilde{B}(\mathcal{E}_M, \mathcal{D}, A) = B^{\circ}(\mathcal{E}_M, D, A) \qquad (D(m) = \mathcal{D}(m,1))$$

Note on monoidality in manifold variable

$$\mathcal{E}_M(m) \times \mathcal{E}_N(n) \to \mathcal{E}_{M \amalg N}(m+n) \implies B(\mathcal{E}_M, \mathcal{D}, A) \wedge B(\mathcal{E}_N, \mathcal{D}, A) \to B(\mathcal{E}_{M\amalg N}, \mathcal{D}, A)$$

Monoidal but not symmetric

Smaller bar constructions

Bar construction $\tilde{B}(\mathcal{E}_M, \mathcal{D}, A)$

$$\tilde{B}_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k) \quad \overset{\circlearrowleft}{\underset{k}{\bullet}} \quad A^{(k)}$$

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$$\tilde{B}_2 = \bigvee_{k_0,k_1,k_2} (\mathcal{E}_M(k_2) \underset{\Sigma_{k_2}}{\times} \mathcal{D}(k_1,k_2) \underset{\Sigma_{k_1}}{\times} \mathcal{D}(k_0,k_1))_+ \underset{\Sigma_{k_0}}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_2) \quad \overset{\circlearrowleft}{\underset{k_2}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_1}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_0}{\bullet}} \quad A^{(k_0)}$$

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$$\tilde{B}(\mathcal{E}_M, \mathcal{D}, A) = B^{\circ}(\mathcal{E}_M, D, A) \qquad (D(m) = \mathcal{D}(m,1))$$

$B(\mathcal{E}, \mathcal{D}, A)$

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Monoidal but not symmetric

$$(\mathcal{E}_M(m) \times \mathcal{E}_N(n)) \underset{\Sigma_m \times \Sigma_n}{\times} \Sigma_{m+n} \xrightarrow{\cong} \mathcal{E}_{M \amalg N}(m+n) \implies \tilde{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \tilde{B}(\mathcal{E}_N, \mathcal{D}, A) \xrightarrow{\cong} \tilde{B}(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

Strong symmetric monoidal

Smaller bar constructions

Bar construction $\bar{B}(\mathcal{E}_M, \mathcal{D}, A)$

$$\bar{B}_0 = \bigvee_{k_0} \mathcal{E}_M(k_0)_+ \underset{\Sigma_{k_0} \wr O(d)}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k) \overset{\circlearrowleft}{\underset{k}{\bullet}} A^{(k)}$$

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$$\bar{B}_2 = \bigvee_{k_0, k_1, k_2} (\mathcal{E}_M(k_2) \underset{\Sigma_{k_2} \wr O(d)}{\times} \mathcal{D}(k_1, k_2) \underset{\Sigma_{k_1} \wr O(d)}{\times} \mathcal{D}(k_0, k_1))_+ \underset{\Sigma_{k_0} \wr O(d)}{\wedge} A^{(k_0)} \qquad \mathcal{E}_M(k_2) \overset{\circlearrowleft}{\underset{k_2}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_1}{\bullet}} \leftarrow \overset{\circlearrowleft}{\underset{k_0}{\bullet}} A^{(k_0)}$$

etc

$$\tilde{B}(\mathcal{E}_M, \mathcal{D}, A) = B^\circ(\mathcal{E}_M, D, A) \qquad (D(m) = \mathcal{D}(m, 1))$$

$$\mathrm{Aut}_{\mathcal{D}}(k) = \Sigma_k \wr O(d) \qquad \Sigma_k \ltimes O(d)^n$$

Note on monoidality in manifold variable

$$\mathcal{E}_M(m) \times \mathcal{E}_N(n) \to \mathcal{E}_{M \amalg N}(m+n) \implies B(\mathcal{E}_M, \mathcal{D}, A) \wedge B(\mathcal{E}_N, \mathcal{D}, A) \to B(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

Monoidal but not symmetric

$$(\mathcal{E}_M(m) \times \mathcal{E}_N(n)) \underset{\Sigma_m \times \Sigma_n}{\times} \Sigma_{m+n} \xrightarrow{\cong} \mathcal{E}_{M \amalg N}(m+n) \implies \tilde{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \tilde{B}(\mathcal{E}_N, \mathcal{D}, A) \xrightarrow{\cong} \tilde{B}(\mathcal{E}_{M \amalg N}, \mathcal{D}, A)$$

Strong symmetric monoidal

Monoidality in the algebra variable

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A \wedge B) \to \bar{B}(\mathcal{E}_M \times \mathcal{E}_M, \mathcal{D} \times \mathcal{D}, A \wedge B) \cong \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \bar{B}(\mathcal{E}_M, \mathcal{D}, B)$$

Not a weak equivalence in general, except in unital case:

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A \wedge B) \to \bar{B}(\mathcal{E}^u_M \times \mathcal{E}^u_M, \mathcal{D}^u \times \mathcal{D}^u, A \wedge B) \cong \bar{B}(\mathcal{E}^u_M, \mathcal{D}^u, A) \wedge \bar{B}(\mathcal{E}^u_M, \mathcal{D}^u, B)$$

How do you prove something like this?

Quillen's Theorem A

Suffices to prove that the comparison map

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^u(m, -) \times \mathcal{D}^u(n, -)) \xrightarrow{\simeq} \mathcal{E}^u_M(m) \times \mathcal{E}^u_M(n)$$

is a weak equivalence.

Let's look at similar but easier comparison

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m, n), -)) \xrightarrow{\simeq} \mathcal{E}^2_M(m, n)$$

where $\mathcal{D}^2((m, n), k) \subset \mathcal{D}^u(m, k) \times \mathcal{D}^u(n, k)$, $\mathcal{E}^2_M(m, n) \subset \mathcal{E}^u_M(m) \times \mathcal{E}^u_M(n)$ (surjective on π_0 maps)

Monoidality in the algebra variable

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Monoidality in the algebra variable

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, A \wedge B) \to \bar{B}(\mathcal{E}_M \times \mathcal{E}_M, \mathcal{D} \times \mathcal{D}, A \wedge B) \cong \bar{B}(\mathcal{E}_M, \mathcal{D}, A) \wedge \bar{B}(\mathcal{E}_M, \mathcal{D}, B)$$

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Monoidality in the algebra variable

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The two-sided bar construction and partitions

Fix M, n and look at $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n, -))$

$B(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-)) \xrightarrow{\simeq} \mathcal{E}_M(n)$

$$\bar{B}_0 = \coprod_{k_0} \mathcal{E}_M(k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0)$$

$$\bar{B}_1 = \coprod_{k_0, k_1} \mathcal{E}_M(k_1) \underset{\Sigma_{k_1} \wr O(d)}{\times} \mathcal{D}(k_0, k_1) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0)$$

$$\vdots \qquad \vdots$$

$$\bar{B}_r = \coprod_{k_0, \dots, k_r} \mathcal{E}_M(k_r) \underset{\Sigma_{k_r} \wr O(d)}{\times} \mathcal{D}(k_{r-1}, k_r) \underset{\Sigma_{k_{r-1}} \wr O(d)}{\times} \cdots \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0)$$

The two-sided bar construction and partitions

Fix M, n and look at $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-))$

$\mathcal{E}(P_\bullet)$

$$\bar{B}_0 = \coprod_{k_0} \mathcal{E}_M(k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0) \qquad P_0$$

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$$\vdots \qquad \vdots$$

$$\bar{B}_r = \coprod_{k_0,\ldots,k_r} \mathcal{E}_M(k_r) \underset{\Sigma_{k_r} \wr O(d)}{\times} \mathcal{D}(k_{r-1}, k_r) \underset{\Sigma_{k_{r-1}} \wr O(d)}{\times} \cdots \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0) \qquad P_r \le \cdots \le P_0$$

For $P_\bullet = (P_r \le \cdots \le P_0)$, define subspace $\mathcal{E}(P_\bullet)$. Can recover $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-))$ from these.

$$\mathcal{E}_M(k_r) \underset{\Sigma_{k_r} \wr O(d)}{\times} \mathcal{D}(k_{r-1}, k_r) \underset{\Sigma_{k_{r-1}} \wr O(d)}{\times} \cdots \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0) = \coprod_{\substack{P_r \le \cdots \le P_0 \\ \#P_i = k_i}} \mathcal{E}(P_r \le \cdots \le P_0)$$

Center point map $\mathcal{E}(P_\bullet) \to \mathcal{E}_M(n) \to C_{Gl(d)}(n, M)$

The two-sided bar construction and partitions

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For $P_\bullet = (P_r \leq \cdots \leq P_0)$, define subspace $\mathcal{E}(P_\bullet)$. Can recover $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-))$ from these.

$$\mathcal{E}_M(k_r) \underset{\Sigma_{k_r} \wr O(d)}{\times} \mathcal{D}(k_{r-1}, k_r) \underset{\Sigma_{k_{r-1}} \wr O(d)}{\times} \cdots \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0) = \coprod_{\substack{P_r \leq \cdots \leq P_0 \\ \#P_i = k_i}} \mathcal{E}(P_r \leq \cdots \leq P_0)$$

Center point map $\mathcal{E}(P_\bullet) \to \mathcal{E}_M(n) \to C_{Gl(d)}(n, M)$

$\mathcal{E}(P_1 \leq P_0) \to \mathcal{E}(P_0)$

The two-sided bar construction and partitions

Fix M, n and look at $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n, -))$

$$\bar{B}_0 = \coprod_{k_0} \mathcal{E}_M(k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}(n, k_0) \qquad P_0$$

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Choose a nice Riemannian metric on M
(Convexity radius ≥ 1, sectional curvatures between -1 and 1)

Choose a large parameter λ to use for scale

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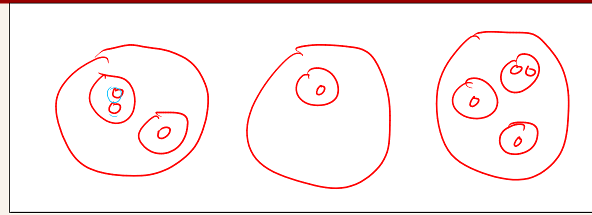

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The two-sided bar construction and the Čech complex

Definition of $U(P_\bullet)$

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Observations

- Every element of $C_{Gl(d)}(n, M)$ is in some $U(P)$
- $U(P_\bullet) = U(P_r) \cap \cdots \cap U(P_0)$
- $U(P_0) \cap U(P_1)$ is non-empty if and only if either $P_0 \leq P_1$ or $P_1 \leq P_0$

In other words:

- $\{U(P)\}$ is an open cover of $C_{Gl(d)}(n, M)$ and
- $\coprod_{P_r \leq \cdots \leq P_0} U(P_\bullet)$ is the rth level of its Čech nerve

On geometric realization we get

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n, -)) = |\coprod \mathcal{E}(P_\bullet)| \xleftarrow{\simeq} |\coprod E(P_\bullet)| \xrightarrow{\simeq} |\coprod U(P_\bullet)|$$

This "proves" center point map $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n, -)) \to \mathcal{E}_M(n)$ is an equivalence

(Easier to prove this by other methods)

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On geometric realization we get

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-)) = |\coprod \mathcal{E}(P_\bullet)| \xleftarrow{\simeq} |\coprod E(P_\bullet)| \xrightarrow{\simeq} |\coprod U(P_\bullet)|$$

This “proves” center point map $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}(n,-)) \to \mathcal{E}_M(n)$ is an equivalence

(Easier to prove this by other methods)

The two-sided bar construction and the Čech complex

Definition of $U(P_\bullet)$

- If i, j are in same block of P_ℓ, then $d(x_i, x_j) < \lambda^{-2k_\ell}$
- If i, j are in different blocks of P_ℓ, then $d(x_i, x_j) > \lambda^{-2k_\ell+1}$

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Example: $\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m,n),-)) \xrightarrow{\simeq} \mathcal{E}^2_M(m,n)$

$$\bar{B}_0 = \coprod_{k_0} \mathcal{E}_M(k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}^2((m,n),k_0)$$

$$\bar{B}_1 = \coprod_{k_0,k_1} \mathcal{E}_M(k_1) \underset{\Sigma_{k_1} \wr O(d)}{\times} \mathcal{D}(k_1,k_0) \underset{\Sigma_{k_0} \wr O(d)}{\times} \mathcal{D}((m,n),k_0)$$

$$\vdots \qquad \vdots$$

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Now partitions of $m+n$

Define $U(P_\bullet) \subset C_{Gl(d)}(m,M) \times C_{Gl(d)}(n,M)$ by exactly the same distance conditions

Define $E(P_\bullet)$ by exactly the same radius and center point conditions
(But now outer little disks are in $\mathcal{D}^2((m,n),k_0)$, so the first m disks can overlap with the last n)

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$$|\coprod U(P_\bullet)| \xrightarrow{\simeq} C^2_{Gl(d)}((m,n),M), \qquad \bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m,n),-)) \to \mathcal{E}^2_M(m,n) \xrightarrow{\simeq} C^2_{Gl(d)}((m,n),M)$$

$$C^2_{Gl(d)}((m,n),M) \subset C_{Gl(d)}(m,M) \times C_{Gl(d)}(n,M)$$

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$$|\coprod U(P_\bullet)| \downarrow \simeq$$

$$\bar{B}(\mathcal{E}_M, \mathcal{D}, \mathcal{D}^2((m,n),-)) \longrightarrow \mathcal{E}^2_M(m,n) \xrightarrow[\simeq]{} C^2_{Gl(d)}((m,n),M)$$

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 & \searrow & & & \downarrow \simeq \\
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