Localization Sequences in *THH* and *TC*

Michael A. Mandell

Indiana University

Algebraic Topology Conference in Buenos Aires
November 12, 2008

Overview

Localization Sequences in *THH* and *TC*

- Joint work with Andrew Blumberg

Overview

Localization Sequences in *THH* and *TC*

- Joint work with Andrew Blumberg
- Preprint: arXiv:0802.3938 [math.KT]

Overview

Localization Sequences in *THH* and *TC*

- Joint work with Andrew Blumberg
- Preprint: arXiv:0802.3938 [math.KT]

Goal: Prove the analogue of the Thomason-Trobaugh K-theory Mayer-Vietoris and localization theorems in *THH* and *TC*

Overview

Localization Sequences in *THH* and *TC*

- Joint work with Andrew Blumberg
- Preprint: arXiv:0802.3938 [math.KT]

Goal: Prove the analogue of the Thomason-Trobaugh K-theory Mayer-Vietoris and localization theorems in *THH* and *TC*

Strategy: Prove the analogue of Keller's Hochschild homology and cyclic homology localization theorems for *THH* and *TC*

Overview

Localization Sequences in *THH* and *TC*

- Joint work with Andrew Blumberg
- Preprint: arXiv:0802.3938 [math.KT]

Goal: Prove the analogue of the Thomason-Trobaugh K-theory Mayer-Vietoris and localization theorems in *THH* and *TC*

Strategy: Prove the analogue of Keller's Hochschild homology and cyclic homology localization theorems for *THH* and *TC*

Recurring theme: connective vs. non-connective ring spectra

Overview

Localization Sequences in *THH* and *TC*

- Joint work with Andrew Blumberg
- Preprint: arXiv:0802.3938 [math.KT]

Goal: Prove the analogue of the Thomason-Trobaugh K-theory Mayer-Vietoris and localization theorems in *THH* and *TC*

Strategy: Prove the analogue of Keller's Hochschild homology and cyclic homology localization theorems for *THH* and *TC*

Ocurring theme: connective vs. non-connective ring spectra

Motivation

Hesselholt and Madsen: Conjectured “Additive Motivic Spectral Sequence”

- Abut to variant of *TR*
- Edge homomorphism from De Rham–Witt complex
- (“Homotopy invariant” *TR* is contractible)

Motivation

Hesselholt and Madsen: Conjectured “Additive Motivic Spectral Sequence”

- ~~Abut~~ converge to variant of *TR*
- Edge homomorphism from De Rham–Witt complex
- (“Homotopy invariant” *TR* is contractible)

Theory of Levine on motivic spectral seqs

Motivation

Hesselholt and Madsen: Conjectured "Additive Motivic Spectral Sequence"

- Abut to variant of *TR*
- Edge homomorphism from De Rham–Witt complex
- ("Homotopy invariant" *TR* is contractible)

Motivation

Why did this paper get bumped up ahead of the others in the long list of papers still to write?

Hesselholt and Madsen: Conjectured “Additive Motivic Spectral Sequence”

- Abut to variant of TR
- Edge homomorphism from De Rham–Witt complex
- (“Homotopy invariant” TR is contractible)

Motivation

Why did this paper get bumped up ahead of the others in the long list of papers still to write?

Cortiñas, Haesemeyer, Schlichting, and Weibel:
K-theory and singularities. Proved:

- Weibel conjecture: $K_{-n}X = 0$ for $n > \dim(X)$
- Vorst conjecture: $K_{\dim(R)}X = 0$ for $n > \dim(X)$

Over fields of characteristic zero using Mayer Vietoris and localization in negative cyclic homology.

TC results to extend to fields of characteristic *p*. (?)

Motivation

Why did this paper get bumped up ahead of the others in the long list of papers still to write?

Cortiñas, Haesemeyer, Schlichting, and Weibel:
K-theory and singularities. Proved:

- Weibel conjecture: $K_{-n}X = 0$ for $n > \dim(X)$
- Vorst conjecture: $K_{\dim(R)}X \neq 0$ for $n > \dim(X)$

Over fields of characteristic zero using Mayer Vietoris and localization in negative cyclic homology.

TC results to extend to fields of characteristic *p*. (?)

negative cyclic

Chern character
trace $K \to HN$

Motivation

Why did this paper get bumped up ahead of the others in the long list of papers still to write?

Cortiñas, Haesemeyer, Schlichting, and Weibel:
K-theory and singularities. Proved:

- Weibel conjecture: $K_{-n}X = 0$ for $n > \dim(X)$
- Vorst conjecture: $K_{\dim(R)}X = 0$ for $n > \dim(X)$

Over fields of characteristic zero using Mayer Vietoris and localization in negative cyclic homology.

TC results to extend to fields of characteristic *p*. (?)

Motivation

Why did this paper get bumped up ahead of the others in the long list of papers still to write?

Cortiñas, Haesemeyer, Schlichting, and Weibel:
K-theory and singularities. Proved:

- Weibel conjecture: $K_{-n}X = 0$ for $n > \dim(X)$
- Vorst conjecture: $K_{\dim(R)}X = 0$ for $n > \dim(X)$

Over fields of characteristic zero using Mayer Vietoris and localization in negative cyclic homology.

TC results to extend to fields of characteristic *p*. (?)

Rumor: Geisser and Hesselholt proved Weibel conjecture in characteristic *p*. (?)

Hochschild Homology

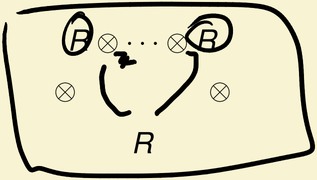
top

Cyclic bar construction

$$N^{cy}_q R = \underbrace{R \otimes \cdots \otimes R \otimes R}_{q \text{ factors}}$$

cyclic ~~simplicial~~ abelian gp

(R DGA simplicial chain cx)



$HH_* R$ is the homology of the resulting chain complex

HH(R) resulting chain complex
"shukla" corresponding spectrum

Topological Hochschild Homology

Cyclic bar construction

$$N^{cy}_q HR = \underbrace{HR \wedge \cdots \wedge HR}_{q \text{ factors}} \wedge HR$$

$$\begin{array}{ccc} & HR \otimes \cdots \otimes HR & \\ \wedge & & \wedge \\ & HR & \end{array}$$

$THH(R)$ is the resulting spectrum

Morita Invariance

Both *HH* and *THH* have Morita invariance:

$$HH(R) \simeq HH(M_n R)$$
$$THH(R) \simeq THH(M_n R)$$

$\Longrightarrow$ Dennis and cyclotomic trace maps

Dennis Trace / Cyclotomic Trace

Map $BGL_nR \to B^{cy}GL_nR$:

Bar const.

$$B_q(GL_nR) = \underbrace{GL_nR \times \cdots \times GL_nR}_{q \text{ factors}}$$

$$B^{cy}_q(GL_nR) = \underbrace{GL_nR \times \cdots \times GL_nR}_{q \text{ factors}} \times GL_nR$$

by $(g_1|\cdots|g_q) \mapsto (g_1|\cdots|g_q)g_q^{-1}\cdots g_1^{-1}$

$g_0 \langle g_1 | \cdots | g_q \rangle$

Map $B^{cy}GL_nR \to N^{cy}(M_nR)$ or to $N^{cy}_q(HM_nR)$.

$$N^{cy}_q(M_nR) = \underbrace{M_nR \otimes \cdots \otimes M_nR}_{q \text{ factors}} \otimes M_nR$$

Fit together to a map $KR = (\coprod BGL_nR)^+ \to THH(R) \to HH(R)$.

Dennis Trace / Cyclotomic Trace

Map $BGL_nR \to B^{cy}GL_nR$:

$$B_q(GL_nR) = \underbrace{GL_nR \times \cdots \times GL_nR}_{q \text{ factors}}$$

$$B_q^{cy}(GL_nR) = \underbrace{GL_nR \times \cdots \times GL_nR}_{q \text{ factors}} \times GL_nR$$

by $(g_1|\cdots|g_q) \mapsto (g_1|\cdots|g_q)g_q^{-1}\cdots g_1^{-1}$.

Map $B^{cy}GL_nR \to N^{cy}(M_nR)$ or to $N_q^{cy}(HM_nR)$.

$$N_q^{cy}(M_nR) = \underbrace{M_nR \otimes \cdots \otimes M_nR}_{q \text{ factors}} \otimes M_nR$$

Fit together to a map $KR = (\coprod BGL_nR)^+ \to THH(R) \to HH(R)$.

Dennis Trace / Cyclotomic Trace

Map $BGL_nR \to B^{cy}GL_nR$:

$$B_q(GL_nR) = \underbrace{GL_nR \times \cdots \times GL_nR}_{q \text{ factors}}$$

$$B^{cy}_q(GL_nR) = \underbrace{GL_nR \times \cdots \times GL_nR}_{q \text{ factors}} \times GL_nR$$

by $(g_1|\cdots|g_q) \mapsto (g_1|\cdots|g_q)g_q^{-1}\cdots g_1^{-1}$.

Map $B^{cy}GL_nR \to N^{cy}(M_nR)$ or to $N^{cy}_q(HM_nR)$.

$$N^{cy}_q(M_nR) = \underbrace{M_nR \otimes \cdots \otimes M_nR}_{q \text{ factors}} \otimes M_nR$$

Fit together to a map $KR = (\coprod BGL_nR)^+ \to THH(R) \to HH(R)$.

Why *THH*?

- Relatively easy to compute
- Stabilization of K-theory [Dundas-McCarthy]

But. . .

Why *THH*?

- Relatively easy to compute
- Stabilization of *K*-theory [Dundas-McCarthy]

But…

Why *THH*?

- Relatively easy to compute
- Stabilization of K-theory [Dundas-McCarthy]

(in sense of Goodwillie calculus)

THH in place of K

↔

stable hty in place of hty)

But…

Why *THH*?

- Relatively easy to compute
- Stabilization of K-theory [Dundas-McCarthy]

But. . .

Not really that close to K-theory

Why *THH*?

- Relatively easy to compute
- Stabilization of K-theory [Dundas-McCarthy]

But…

Not really that close to K-theory

$\Longrightarrow$ HN and TC

Negative Cyclic and *TC*

- Built from *HH* and *THH*
- Still reasonably computable
- Goodwillie: Relative *HN* is rationally equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative *TC* is p-equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).

Negative Cyclic and *TC*

- Built from *HH* and *THH* not in same way
- Still reasonably computable
- Goodwillie: Relative *HN* is rationally equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative *TC* is *p*-equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).

Negative Cyclic and *TC*

- Built from *HH* and *THH*
- Still reasonably computable
- Goodwillie: Relative *HN* is rationally equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative *TC* is *p*-equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).

Negative Cyclic and *TC*

- Built from *HH* and *THH*
- Still reasonably computable
- Goodwillie: Relative *HN* is rationally equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative *TC* is p-equivalent to relative *K*-theory (for surjective maps with nilpotent kernel).

Homotopy cartesian square

$R \to R'$ surj, nilp Kernel

$$\begin{array}{ccc} K(R)_{\mathbb{Q}} & \longrightarrow & K(R')_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ HN(R \otimes \mathbb{Q}) & \longrightarrow & HN(R' \otimes \mathbb{Q}) \end{array}$$

Negative Cyclic and *TC*

- Built from *HH* and *THH*
- Still reasonably computable
- Goodwillie: Relative *HN* is rationally equivalent to relative K-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative *TC* is p-equivalent to relative K-theory (for surjective maps with nilpotent kernel).

Homotopy cartesian square

$$\begin{array}{ccc} K(R)_{\mathbb{Q}} & \longrightarrow & K(R')_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ HN(R \otimes \mathbb{Q}) & \longrightarrow & HN(R' \otimes \mathbb{Q}) \end{array} \qquad \begin{array}{ccc} K(R)^{\wedge}_{p} & \longrightarrow & K(R')^{\wedge}_{p} \\ \downarrow & & \downarrow \\ TC(R, p) & \longrightarrow & TC(R', p) \end{array}$$

Negative Cyclic and *TC*

- Built from *HH* and *THH*
- Still reasonably computable
- Goodwillie: Relative *HN* is rationally equivalent to relative K-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative *TC* is p-equivalent to relative K-theory (for surjective maps with nilpotent kernel).
- Dundas: Generalized p-equivalence to maps of ring spectra.

Homotopy cartesian square

$$\begin{array}{ccc} K(R)_{\mathbb{Q}} & \longrightarrow & K(R')_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ HN(R\otimes\mathbb{Q}) & \longrightarrow & HN(R'\otimes\mathbb{Q}) \end{array} \qquad \begin{array}{ccc} K(R)^{\wedge}_{p} & \longrightarrow & K(R')^{\wedge}_{p} \\ \downarrow & & \downarrow \\ TC(R,p) & \longrightarrow & TC(R',p) \end{array}$$

Quillen's K-Theory of Schemes

$K(X) = K$-theory of exact category of vector bundles on X.

Algebraic-geometric Remarks:

- Definitively correct for quasi-projective varieties.
- "Obviously" not necessarily correct for varieties that do not enough vector bundles.
- Formulation of perfect complex: A complex of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to bounded complex of vector bundles.
- Bounded derived category = full subcategory of derived category of $\mathcal{O}_X$-modules consisting of the perfect complexes.
- Bounded derived category is also the full subcategory of compact objects. [An alternative characterization of perfect complexes.]

Quillen's *K*-Theory of Schemes

$K(X)$ = K-theory of exact category of vector bundles on X.

Algebraic-geometric Remarks:

- Definitively correct for quasi-projective varieties.
- "Obviously" not necessarily correct for varieties that do not enough vector bundles. [SGA6 IV§2] K_0 is $K^\bullet_{\text{naif}}$.
- Formulation of perfect complex: A complex of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to bounded complex of vector bundles.
- Bounded derived category = full subcategory of derived category of $\mathcal{O}_X$-modules consisting of the perfect complexes.
- Bounded derived category is also the full subcategory of compact objects. [An alternative characterization of perfect complexes.]

Quillen's K-Theory of Schemes

$K(X)$ = K-theory of exact category of vector bundles on X.

Algebraic-geometric Remarks:

- Definitively correct for quasi-projective varieties.
- "Obviously" not necessarily correct for varieties that do not enough vector bundles. [SGA6 IV§2] K_0 is $K^{\bullet}_{\text{naif}}$.
- Formulation of perfect complex: A complex of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to bounded complex of vector bundles.
- Bounded derived category = full subcategory of derived category of $\mathcal{O}_X$-modules consisting of the perfect complexes.
- Bounded derived category is also the full subcategory of compact objects. [An alternative characterization of perfect complexes.]

Quillen's K-Theory of Schemes

$K(X) = K$-theory of exact category of vector bundles on X.

Algebraic-geometric Remarks:

- Definitively correct for quasi-projective varieties.
- "Obviously" not necessarily correct for varieties that do not enough vector bundles. [SGA6 IV§2] K_0 is $K^{\bullet}_{\text{naif}}$.
- Formulation of perfect complex: A complex of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to bounded complex of vector bundles.
- Bounded derived category = full subcategory of derived category of $\mathcal{O}_X$-modules consisting of the perfect complexes.
- Bounded derived category is also the full subcategory of compact objects. [An alternative characterization of perfect complexes.]

$$\mathcal{D}(X, \bigoplus_{\text{inf}} Y_\alpha) \xleftarrow{\cong} \bigoplus_{\text{inf}} \mathcal{D}(X, Y_\alpha)$$

Thomason's K-Theory of Schemes

& Trobaugh   Groth Fest III

Basic Outline

- Apply Waldhausen's construction, which generalizes Quillen's from exact categories to categories with cofibrations and weak equivalences
- Work with Complicial biWaldhausen categories and functors: Subcategories of categories of complexes on abelian categories (with restrictions); functors induced by additive functors on the underlying abelian categories.

This gives a K-theory of derived categories (of sorts):

Theorem. If a complicial functor between complicial biWaldhausen categories induces an equivalence of derived categories, it induces an equivalence of K-theory.

Thomason's K-Theory of Schemes

Basic Outline

- Apply Waldhausen's construction, which generalizes Quillen's from exact categories to categories with cofibrations and weak equivalences
- Work with Complicial biWaldhausen categories and functors: Subcategories of categories of complexes on abelian categories (with restrictions); functors induced by additive functors on the underlying abelian categories.

This gives a K-theory of derived categories (of sorts):

Theorem. If a complicial functor between complicial biWaldhausen categories induces an equivalence of derived categories, it induces an equivalence of K-theory.

morally K-thy depends only on derived cat

Problem: Actual functor

Thomason's K-Theory of Schemes

Consequence: Any subcategory (with restrictions) of perfect complexes of $\mathcal{O}_X$-modules whose derived category is the bounded derived category produces the same K-theory.

Flat

$K(X)$ = K-theory of the category of perfect complexes on X.

Variant: For Y a closed subset of X

$K(X \text{ on } Y)$ = K-theory of the category of perfect complexes on X that are supported on Y (acyclic off Y).

Thomason's K-Theory of Schemes

Consequence: Any subcategory (with restrictions) of perfect complexes of $\mathcal{O}_X$-modules whose derived category is the bounded derived category produces the same K-theory.

$K(X) = K$-theory of the category of perfect complexes on X.

Variant: For Y a closed subset of X

$K(X \text{ on } Y) = K$-theory of the category of perfect complexes on X that are supported on Y (acyclic off Y).

Thomason's K-Theory of Schemes

Consequence: Any subcategory (with restrictions) of perfect complexes of $\mathcal{O}_X$-modules whose derived category is the bounded derived category produces the same K-theory.

$K(X)$ = K-theory of the category of perfect complexes on X.

Variant: For Y a closed subset of X $\qquad U = X - Y$

$K(X \text{ on } Y)$ = K-theory of the category of perfect complexes on X that are supported on Y (acyclic off Y).

Thomason Trobaugh Localization Theorem

Localization Theorem. Let $U \subset X$ be open, $Y = X - U$. There is a long exact sequence

$$\cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(U) \to \cdots$$
$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U)$$

Mayer-Vietoris Theorem. Let $U, V \subset X$ with $X = U \cup V$. There is a long exact sequence

$$\cdots \to K_n(U \cap V) \to K_n(U) \oplus K_n(V) \to K_n(X) \to \cdots$$
$$\cdots \to K_0(U \cap V) \to K_0(U) \oplus K_0(V) \to K_0(X)$$

Thomason Trobaugh Localization Theorem

Localization Theorem. Let $U \subset X$ be open, $Y = X - U$. There is a long exact sequence

$$\cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(U) \to \cdots$$
$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U)$$

Mayer-Vietoris Theorem. Let $U, V \subset X$ with $X = U \cup V$. There is a long exact sequence

$$\cdots \to K_n(U \cap V) \to K_n(U) \oplus K_n(V) \to K_n(X) \to \cdots$$
$$\cdots \to K_0(U \cap V) \to K_0(U) \oplus K_0(V) \to K_0(X)$$

Proof

Waldhausen: Fibration sequences for “weaker” weak equivalences.

$\mathcal{C}$ category with a weak equivalences w, and another collection of weak equivalences v with $v \subset w$.

$\mathcal{C}^w$ = the subcategory of $\mathcal{C}$ of objects w-equivalent to the trivial object.

Theorem. (Waldhausen localization sequence)
The following square is homotopy cartesian:

$$\begin{array}{ccc} K(\mathcal{C}^w, v) & \longrightarrow & K(\mathcal{C}^w, w) \\ \downarrow & & \downarrow \\ K(\mathcal{C}, v) & \longrightarrow & K(\mathcal{C}, w) \end{array}$$

$K(\mathcal{C}^w, w)$ is trivial.

Proof

Waldhausen: Fibration sequences for “weaker” weak equivalences.

$\mathcal{C}$ category with a weak equivalences w, and another collection of weak equivalences v with $v \subset w$.

$\mathcal{C}^w$ = the subcategory of $\mathcal{C}$ of objects w-equivalent to the trivial object.

Theorem. (Waldhausen localization sequence)
The following square is homotopy cartesian:

$$\begin{array}{ccc} K(\mathcal{C}^w, v) & \longrightarrow & K(\mathcal{C}^w, w) \\ \downarrow & & \downarrow \\ K(\mathcal{C}, v) & \longrightarrow & K(\mathcal{C}, w) \end{array}$$

$K(\mathcal{C}^w, w)$ is trivial.

Proof

Waldhausen: Fibration sequences for “weaker” weak equivalences.

$\mathcal{C}$ category with a weak equivalences w, and another collection of weak equivalences v with $v \subset w$.

$\mathcal{C}^w$ = the subcategory of $\mathcal{C}$ of objects w-equivalent to the trivial object.

Theorem. (Waldhausen localization sequence)
The following square is homotopy cartesian:

$$\begin{array}{ccc} K(\mathcal{C}^w, v) & \longrightarrow & K(\mathcal{C}^w, w) \\ \downarrow & & \downarrow \\ K(\mathcal{C}, v) & \longrightarrow & K(\mathcal{C}, w) \end{array}$$

$K(\mathcal{C}^w, w)$ is trivial.

Proof

Take $\mathcal{C}$ = Perfect complexes on X

v = Quasi-isomorphisms
w = Maps that are quasi-isomorphisms on U
$v \subset w$

$\mathcal{C}^w$ = Perfect complexes supported on Y.

Thomason and Trobaugh prove that the derived category $\mathcal{D}(\mathcal{C}, w)$ is cofinal in the bounded derived category of U.

Proof

Take $\mathcal{C}$ = Perfect complexes on X

v = Quasi-isomorphisms
w = Maps that are quasi-isomorphisms on U
$v \subset w$

$\mathcal{C}^w$ = Perfect complexes supported on Y.

Thomason and Trobaugh prove that the derived category $\mathcal{D}(\mathcal{C}, w)$ is cofinal in the bounded derived category of U.

Proof

Take $\mathcal{C}$ = Perfect complexes on X

v = Quasi-isomorphisms
w = Maps that are quasi-isomorphisms on U
$v \subset w$

$\mathcal{C}^w$ = Perfect complexes supported on Y.

Thomason and Trobaugh prove that the derived category $\mathcal{D}(\mathcal{C}, w)$ is cofinal in the bounded derived category of U.

Proof

Take $\mathcal{C}$ = Perfect complexes on X

v = Quasi-isomorphisms
w = Maps that are quasi-isomorphisms on U
$v \subset w$

$\mathcal{C}^w$ = Perfect complexes supported on Y.

Thomason and Trobaugh prove that the derived category $\mathcal{D}(\mathcal{C}, w)$ is cofinal in the bounded derived category of U.

Neeman's Abstract Reformulation

Let $\mathcal{S}$ be a triangulated category generated by its compact objects $\mathcal{S}^c$
and is closed under small coproducts
and assume $\mathcal{S}^c$ is small.

Let $\mathcal{R}$ be localizing subcategory gen. by some set of compact objects.

Let $\mathcal{T}$ be the triangulated quotient $\mathcal{S}/\mathcal{R}$.

This means $\mathcal{T}$ is the localization of $\mathcal{S}$ with respect to the maps whose cofibers are in $\mathcal{R}$.

Theorem

- The compact objects $\mathcal{S}^c$ of $\mathcal{S}$ map to compact objects of $\mathcal{T}$.
- The induced functor $\mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful and $\mathcal{T}^c$ is the thick subcategory generated by its image.

T-T localization set-up can be reformulated in terms of quotients of triangulated categories.

Neeman's Abstract Reformulation

Let $\mathcal{S}$ be a triangulated category generated by its compact objects $\mathcal{S}^c$
and is closed under small coproducts
and assume $\mathcal{S}^c$ is small.

Let $\mathcal{R}$ be localizing subcategory gen. by some set of compact objects.

Let $\mathcal{T}$ be the triangulated quotient $\mathcal{S}/\mathcal{R}$.

This means $\mathcal{T}$ is the localization of $\mathcal{S}$ with respect to the maps whose cofibers are in $\mathcal{R}$.

Theorem

- The compact objects $\mathcal{S}^c$ of $\mathcal{S}$ map to compact objects of $\mathcal{T}$.
- The induced functor $\mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful and $\mathcal{T}^c$ is the thick subcategory generated by its image.

T-T localization set-up can be reformulated in terms of quotients of triangulated categories.

Neeman's Abstract Reformulation

Let $\mathcal{S}$ be a triangulated category generated by its compact objects $\mathcal{S}^c$
and is closed under small coproducts
and assume $\mathcal{S}^c$ is small.
Let $\mathcal{R}$ be localizing subcategory gen. by some set of compact objects.
Let $\mathcal{T}$ be the triangulated quotient $\mathcal{S}/\mathcal{R}$.

This means $\mathcal{T}$ is the localization of $\mathcal{S}$ with respect to the maps whose cofibers are in $\mathcal{R}$.

Theorem

- The compact objects $\mathcal{S}^c$ of $\mathcal{S}$ map to compact objects of $\mathcal{T}$.
- The induced functor $\mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful and $\mathcal{T}^c$ is the thick subcategory generated by its image.

T-T localization set-up can be reformulated in terms of quotients of triangulated categories.

Remark on Cofinality

Cofinality implies iso on K_n for $n > 0$ but only an injection on K_0. This is why the localization sequence is generally not surjective on K_0:

$$\cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(U) \to \cdots$$
$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U)$$

The sequence actually continues with the Bass negative K-groups:

$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U) \to K_{-1}(X \text{ on } Y) \to K_{-1}(X) \to \cdots$$

These groups are defined inductively by

$$K_{-n-1}X = \operatorname{Coker}\left(K_{-n}(X \times \operatorname{Spec}\mathbb{Z}[x]) \oplus K_{-n}(X \times \operatorname{Spec}\mathbb{Z}[x^{-1}]) \to K_{-n}(X \times \operatorname{Spec}\mathbb{Z}[x, x^{-1}])\right)$$

Remark on Cofinality

Cofinality implies iso on K_n for $n > 0$ but only an injection on K_0. This is why the localization sequence is generally not surjective on K_0:

$$\cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(U) \to \cdots$$
$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U)$$

The sequence actually continues with the Bass negative K-groups:

$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U) \to K_{-1}(X \text{ on } Y) \to K_{-1}(X) \to \cdots$$

These groups are defined inductively by

$$K_{-n-1}X = \mathrm{Coker}\,(K_{-n}(X \times \mathrm{Spec}\,\mathbb{Z}[x]) \oplus K_{-n}(X \times \mathrm{Spec}\,\mathbb{Z}[x^{-1}]) \to K_{-n}(X \times \mathrm{Spec}\,\mathbb{Z}[x, x^{-1}]))$$

Remark on Cofinality

Cofinality implies iso on K_n for $n > 0$ but only an injection on K_0. This is why the localization sequence is generally not surjective on K_0:

$$\cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(U) \to \cdots$$
$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U)$$

The sequence actually continues with the Bass negative K-groups:

$$\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U) \to K_{-1}(X \text{ on } Y) \to K_{-1}(X) \to \cdots$$

These groups are defined inductively by

$K_{-n}(R[x]) \oplus K_{-n}(R[x^{-1}])$
$\downarrow$
$K_{-n}(R[x, x^{-1}])$

$$K_{-n-1}X = \operatorname{Coker}\big(K_{-n}(X \times \operatorname{Spec}\mathbb{Z}[x]) \oplus K_{-n}(X \times \operatorname{Spec}\mathbb{Z}[x^{-1}]) \to K_{-n}(X \times \operatorname{Spec}\mathbb{Z}[x, x^{-1}])\big)$$

Thomason Trobaugh Bass K-Theory Spectrum

Thomason and Trobaugh construct a non-connective K-theory spectrum $K^B X$ essentially by doing Bass' algebraic construction on the spectrum level.

In terms of K^B, the localization theorem asserts a cofiber sequence of spectra

$$K^B(X \text{ on } Y) \to K^B(X) \to K(U)$$

and the Mayer-Vietoris theorem asserts a cofiber sequence of spectra

$$K^B(U \cap V) \to K^B(U) \vee K^B(V) \to K^B(U \cup V)$$

Thomason Trobaugh Bass K-Theory Spectrum

Thomason and Trobaugh construct a non-connective K-theory spectrum $K^B X$ essentially by doing Bass' algebraic construction on the spectrum level.

In terms of K^B, the localization theorem asserts a cofiber sequence of spectra

$$K^B(X \text{ on } Y) \to K^B(X) \to K(U)$$

and the Mayer-Vietoris theorem asserts a cofiber sequence of spectra

$$K^B(U \cap V) \to K^B(U) \vee K^B(V) \to K^B(U \cup V)$$

Thomason Trobaugh Bass K-Theory Spectrum

Thomason and Trobaugh construct a non-connective K-theory spectrum $K^B X$ essentially by doing Bass' algebraic construction on the spectrum level.

In terms of K^B, the localization theorem asserts a cofiber sequence of spectra

$$K^B(X \text{ on } Y) \to K^B(X) \to K^B(U)$$

and the Mayer-Vietoris theorem asserts a cofiber sequence of spectra

$$K^B(U \cap V) \to K^B(U) \vee K^B(V) \to K^B(U \cup V)$$

First Try: Hochschild-Mitchell construction

For an additive category $\mathcal{C}$

Ring add Cat w/ one object

$$N_q^{cy}\mathcal{C} = \bigoplus_{x_0,\ldots,x_q\in\mathcal{C}} \mathcal{C}(x_q, x_{q-1}) \otimes \cdots \otimes \mathcal{C}(x_1, x_0) \otimes \mathcal{C}(x_0, x_q)$$

Constructs $HH(\mathcal{C})$.

Using (bar construction) Eilenberg-Mac Lane spectra, we get a *spectral category* $\mathcal{C}^{\mathcal{S}}$.

$$N_q^{cy}\mathcal{C}^{\mathcal{S}} = \bigvee_{x_0,\ldots,x_q\in\mathcal{C}} \mathcal{C}(x_q, x_{q-1}) \wedge \cdots \wedge \mathcal{C}(x_1, x_0) \wedge \mathcal{C}(x_0, x_q)$$

Constructs $THH(\mathcal{C})$.

We could apply this to the category of vector bundles.

First Try: Hochschild-Mitchell construction

For an additive category $\mathcal{C}$

$$N^{cy}_q \mathcal{C} = \bigoplus_{x_0,\dots,x_q \in \mathcal{C}} \mathcal{C}(x_q, x_{q-1}) \otimes \cdots \otimes \mathcal{C}(x_1, x_0) \otimes \mathcal{C}(x_0, x_q)$$

Constructs $HH(\mathcal{C})$.

Using (bar construction) Eilenberg-Mac Lane spectra, we get a *spectral category* $\mathcal{C}^{\mathcal{S}}$.

$$N^{cy}_q \mathcal{C}^{\mathcal{S}} = \bigvee_{x_0,\dots,x_q \in \mathcal{C}} \mathcal{C}(x_q, x_{q-1}) \wedge \cdots \wedge \mathcal{C}(x_1, x_0) \wedge \mathcal{C}(x_0, x_q)$$

Constructs $THH(\mathcal{C})$.

We could apply this to the category of vector bundles.

First Try: Hochschild-Mitchell construction

For an additive category $\mathcal{C}$

$$N_q^{cy}\mathcal{C} = \bigoplus_{x_0,\ldots,x_q\in\mathcal{C}} \mathcal{C}(x_q,x_{q-1})\otimes\cdots\otimes\mathcal{C}(x_1,x_0)\otimes\mathcal{C}(x_0,x_q)$$

Constructs $HH(\mathcal{C})$.

Using (bar construction) Eilenberg-Mac Lane spectra, we get a *spectral category* $\mathcal{C}^{\mathcal{S}}$.

$$N_q^{cy}\mathcal{C}^{\mathcal{S}} = \bigvee_{x_0,\ldots,x_q\in\mathcal{C}} \mathcal{C}(x_q,x_{q-1})\wedge\cdots\wedge\mathcal{C}(x_1,x_0)\wedge\mathcal{C}(x_0,x_q)$$

Constructs $THH(\mathcal{C})$

We could apply this to the category of vector bundles.

Second Try: Dundas-McCarthy

Invariance problems of Hochschild-Mitchell construction: Treats an exact category $\mathcal{C}$ as an additive category. (Only sees split exact sequences.)

Solution:
Mix Waldhausen's $S_\bullet$-construction in with the Hochschild-Mitchell construction .

Nice consequence:
Can reformulate cyclotomic trace as inclusion of objects in Hochschild-Mitchell construction.

Second Try: Dundas-McCarthy

Invariance problems of Hochschild-Mitchell construction: Treats an exact category $\mathcal{C}$ as an additive category. (Only sees split exact sequences.)

Solution:
Mix Waldhausen's $S_\bullet$-construction in with the Hochschild-Mitchell construction .

Nice consequence:
Can reformulate cyclotomic trace as inclusion of objects in Hochschild-Mitchell construction.

Second Try: Dundas-McCarthy

Invariance problems of Hochschild-Mitchell construction: Treats an exact category $\mathcal{C}$ as an additive category. (Only sees split exact sequences.)

Solution:
Mix Waldhausen's $S_\bullet$-construction in with the Hochschild-Mitchell construction .

Nice consequence:
Can reformulate cyclotomic trace as inclusion of objects in Hochschild-Mitchell construction.

!!!

Nice work!

Mayer-Vietoris

The Dundas-McCarthy construction cannot satisfy Mayer-Vietoris.

Example. Look at the projective (elliptic) curve

$$x_0 x_2^2 = x_1^3 - 3x_0^2 x_1$$

This has an open cover by the affines

$U = \{x_0 \neq 0\} = \operatorname{Spec} \mathbb{Z}[x, y]/(y^2 = x^3 - 3x)$

$V = \{x_2 \neq 0\} = \operatorname{Spec} \mathbb{Z}[u, v]/(u = v^3 - 3u^2 v)$

for $x = x_1/x_0$, $y = x_2/x_0$, $u = x_0/x_2$, $v = x_1/x_2$

Then $U \cap V = \operatorname{Spec} \mathbb{Z}[x, y, y^{-1}]/(y^2 = x^3 - 3x)$, but

$$THH_0(U) \oplus THH_0(V) = \mathbb{Z}[x, y] \oplus \mathbb{Z}[u, v] \to \mathbb{Z}[x, y, y^{-1}] = THH_0(U \cap V)$$

is not surjective. (Here $u \mapsto 1/y$ and $v \mapsto x/y$.)

Mayer-Vietoris

The Dundas-McCarthy construction cannot satisfy Mayer-Vietoris.

Example. Look at the projective (elliptic) curve

$$x_0 x_2^2 = x_1^3 - 3x_0^2 x_1$$

This has an open cover by the affines

$U = \{x_0 \neq 0\} = \operatorname{Spec} \mathbb{Z}[x, y]/(y^2 = x^3 - 3x)$

$V = \{x_2 \neq 0\} = \operatorname{Spec} \mathbb{Z}[u, v]/(u = v^3 - 3u^2 v)$

for $x = x_1/x_0$, $y = x_2/x_0$, $u = x_0/x_2$, $v = x_1/x_2$

Then $U \cap V = \operatorname{Spec} \mathbb{Z}[x, y, y^{-1}]/(y^2 = x^3 - 3x)$, but

$$THH_0(U) \oplus THH_0(V) = \mathbb{Z}[x, y] \oplus \mathbb{Z}[u, v] \to \mathbb{Z}[x, y, y^{-1}] = THH_0(U \cap V)$$

is not surjective. (Here $u \mapsto 1/y$ and $v \mapsto x/y$.)

Mayer-Vietoris

The Dundas-McCarthy construction cannot satisfy Mayer-Vietoris.

Example. Look at the projective (elliptic) curve

$$x_0x_2^2 = x_1^3 - 3x_0^2x_1$$

This has an open cover by the affines

$$U = \{x_0 \neq 0\} = \operatorname{Spec} \mathbb{Z}[x,y]/(y^2 = x^3 - 3x)$$
$$V = \{x_2 \neq 0\} = \operatorname{Spec} \mathbb{Z}[u,v]/(u = v^3 - 3u^2v)$$

for $x = x_1/x_0$, $y = x_2/x_0$, $u = x_0/x_2$, $v = x_1/x_2$

Then $U \cap V = \operatorname{Spec} \mathbb{Z}[x,y,y^{-1}]/(y^2 = x^3 - 3x)$, but

$$THH_0(U) \oplus THH_0(V) = \mathbb{Z}[x,y] \oplus \mathbb{Z}[u,v] \to \mathbb{Z}[x,y,y^{-1}] = THH_0(U \cap V)$$

is not surjective. (Here $u \mapsto 1/y$ and $v \mapsto x/y$.)

comm ring → $THH_0(R) = R$

Bottom level Čech complex for $H^*(X;\mathcal{O}_X)$

$THH_{-1}(E) = H^1(E,\mathcal{O}_E)$

Bass Construction

This is not related to the Bass construction.

No negative Bass *THH* groups for rings:

$$\mathrm{Coker}\left(THH_0(R[x]) \oplus THH_0(R[x^{-1}]) \to THH_0(R[x, x^{-1}])\right)$$

is always surjective. It is

$$R[x] \oplus R[x^{-1}] \to R[x, x^{-1}].$$

This problem has a different source. (Related to non-connective spectra in a different way.)

Bass Construction

This is not related to the Bass construction.

No negative Bass *THH* groups for rings:

$$\mathrm{Coker}\left(THH_0(R[x]) \oplus THH_0(R[x^{-1}]) \to THH_0(R[x,x^{-1}])\right)$$

is always surjective. It is

$$R[x] \oplus R[x^{-1}] \to R[x,x^{-1}].$$

This problem has a different source. (Related to non-connective spectra in a different way.)

Bass Construction

This is not related to the Bass construction.

No negative Bass *THH* groups for rings:

$$\mathrm{Coker}\left(THH_0(R[x]) \oplus THH_0(R[x^{-1}]) \to THH_0(R[x,x^{-1}])\right)$$

is always surjective. It is

$$R \rightarrowtail R[x] \oplus R[x^{-1}] \twoheadrightarrow R[x,x^{-1}].$$

This problem has a different source. (Related to non-connective spectra in a different way.)

Bass Construction

This is not related to the Bass construction.

No negative Bass *THH* groups for rings:

$$\mathrm{Coker}\left(THH_0(R[x]) \oplus THH_0(R[x^{-1}]) \to THH_0(R[x,x^{-1}])\right)$$

is always surjective. It is

$$R[x] \oplus R[x^{-1}] \to R[x,x^{-1}].$$

This problem has a different source. (Related to non-connective spectra in a different way.)

Third Try: Geisser-Hesselholt

THH of rings localizes: $\pi_* THH(R[S^{-1}]) = \pi_* THH(R) \otimes R[S^{-1}]$.

In other words, for a ring $\pi_* THH(R)$ is a quasi-coherent sheaf

Define $THH(X)$ as the Čech spectrum of an affine open cover, or as the hyper-cohomology spectrum.

Tautologically satisfies Mayer-Vietoris.

But no construction of $THH(X \text{ on } Y)$ for localization sequence.

Third Try: Geisser-Hesselholt

THH of rings localizes: $\pi_* THH(R[S^{-1}]) = \pi_* THH(R) \otimes R[S^{-1}]$.

In other words, for a ring $\pi_* THH(R)$ is a quasi-coherent sheaf

Define $THH(X)$ as the Čech spectrum of an affine open cover, or as the hyper-cohomology spectrum.

Tautologically satisfies Mayer-Vietoris.

But no construction of $THH(X \text{ on } Y)$ for localization sequence.

Third Try: Geisser-Hesselholt

THH of rings localizes: $\pi_* THH(R[S^{-1}]) = \pi_* THH(R) \otimes R[S^{-1}]$.

In other words, for a ring $\pi_* THH(R)$ is a quasi-coherent sheaf

Define $THH(X)$ as the Čech spectrum of an affine open cover, or as the hyper-cohomology spectrum.

Tautologically satisfies Mayer-Vietoris.

But no construction of $THH(X \text{ on } Y)$ for localization sequence.

Third Try: Geisser-Hesselholt

THH of rings localizes: $\underbrace{\pi_* THH(R[S^{-1}]) = \pi_* THH(R) \otimes R[S^{-1}]}$.

In other words, for a ring $\pi_* THH(R)$ is a quasi-coherent sheaf

Define $THH(X)$ as the Čech spectrum of an affine open cover, or as the hyper-cohomology spectrum.

Tautologically satisfies Mayer-Vietoris.

But no construction of $THH(X \text{ on } Y)$ for localization sequence.

Third Try: Geisser-Hesselholt

THH of rings localizes: $\pi_* THH(R[S^{-1}]) = \pi_* THH(R) \otimes R[S^{-1}]$.

In other words, for a ring $\pi_* THH(R)$ is a quasi-coherent sheaf

Define $THH(X)$ as the Čech spectrum of an affine open cover, or as the hyper-cohomology spectrum.

Tautologically satisfies Mayer-Vietoris.

But no construction of $THH(X \text{ on } Y)$ for localization sequence.

Construction for *HH*: Keller

Force localization to hold:

Setup: $\mathcal{S}$ a DG-category, $\mathcal{R}$ a subcategory.
E.g., $\mathcal{S}$ a category of complexes, $\mathcal{R}$ the acyclics.
Define: $HH(\mathcal{S}, \mathcal{R})$ as the cofiber of Hochschild-Mitchell constructions

$$HH(\mathcal{S}, \mathcal{R}) = \text{Cofiber}(HH(\mathcal{R}) \to HH(\mathcal{S}))$$

(Definition actually due to Kassel.)

Keller then proves (roughly) that a map $(\mathcal{S}, \mathcal{R}) \to (\mathcal{S}', \mathcal{R}')$ that induces an equivalence on triangulated quotients induces an equivalence on *HH*.

Our Approach: Concepts

Work with spectral categories and use the Hochschild-Mitchell complex (actually, the analogue due to Bokstedt).

A spectral category has an associated *homotopy category* defined by π_0 of the mapping spectra, or graded homotopy category defined by π_* of the mapping spectra.

A *pretriangulated* spectral category is (roughly) a spectral category whose homotopy category is triangulated.

Work of Shipley shows that we can enhance a DG-category into a spectral category.

Our Approach: Concepts

Work with spectral categories and use the Hochschild-Mitchell complex (actually, the analogue due to Bokstedt).

A spectral category has an associated *homotopy category* defined by π_0 of the mapping spectra, or graded homotopy category defined by π_* of the mapping spectra.

A *pretriangulated* spectral category is (roughly) a spectral category whose homotopy category is triangulated.

Work of Shipley shows that we can enhance a DG-category into a spectral category.

Our Approach: Concepts

Work with spectral categories and use the Hochschild-Mitchell complex (actually, the analogue due to Bokstedt).

A spectral category has an associated *homotopy category* defined by π_0 of the mapping spectra, or graded homotopy category defined by π_* of the mapping spectra.

A *pretriangulated* spectral category is (roughly) a spectral category whose homotopy category is triangulated.

Work of Shipley shows that we can enhance a DG-category into a spectral category.

Our Approach: Concepts

Work with spectral categories and use the Hochschild-Mitchell complex (actually, the analogue due to Bokstedt).

A spectral category has an associated *homotopy category* defined by π_0 of the mapping spectra, or graded homotopy category defined by π_* of the mapping spectra.

A *pretriangulated* spectral category is (roughly) a spectral category whose homotopy category is triangulated.

Work of Shipley shows that we can enhance a DG-category into a spectral category.

DK-Invariance

Basic kind of equivalence of spectral categories: Dwyer-Kan equivalence.

A DK-equivalence of spectral categories is a spectral functor that is a weak equivalence on mapping spectra and an equivalence on the homotopy category.

Theorem

A DK-equivalence of spectral categories induces a weak equivalence of this THH.

DK-Invariance

Basic kind of equivalence of spectral categories: Dwyer-Kan equivalence.

A DK-equivalence of spectral categories is a spectral functor that is a weak equivalence on mapping spectra and an equivalence on the homotopy category.

Theorem

A DK-equivalence of spectral categories induces a weak equivalence of this THH.

Morita Invariance / Cofinality

Up to DK-equivalence any (small) spectral category embeds in a pretriangulated spectral category.

We use this to simplify statements

Theorem

Let $\mathcal{C} \subset \mathcal{C}'$ be full subcategories of the pretriangulated spectral category $\mathcal{D}$ with the objects of $\mathcal{C}'$ contained in the thick subcategory generated by the objects of $\mathcal{C}$ (in the triangulated category $\pi_0\mathcal{D}$). Then

$$THH(\mathcal{C}) \to THH(\mathcal{C}')$$

is a weak equivalence.

Localization Theorem

Let $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A}' \subset \mathcal{B}'$ be inclusions of full spectral categories and assume that they are all pretriangulated. Let $f\colon \mathcal{B} \to \mathcal{B}'$ be a spectral functor that restricts to $\mathcal{A} \to \mathcal{A}'$.

Theorem (Abstract localization theorem)

If the induced map of triangulated quotients is an equivalence then the map of cofibers

$$\mathrm{Cofiber}(THH(\mathcal{A}) \to THH(\mathcal{B})) \longrightarrow \mathrm{Cofiber}(THH(\mathcal{A}') \to THH(\mathcal{B}'))$$

is an equivalence.

Conseqences

We can use a spectral model of the derived category of perfect complexes on X to define $THH(X)$

We can use the full subcategory of U-acyclics to define $THH(X \text{ on } Y)$

Theorem (Localization for open subschemes)

There is a cofibration sequence of spectra

$$THH(X \text{ on } Y) \to THH(X) \to THH(U)$$

Theorem (Mayer-Vietoris)

There is a cofibration sequence of spectra

$$THH(U \cap V) \to THH(U) \vee THH(V) \to THH(U \cup V)$$

Conseqences

We can use a spectral model of the derived category of perfect complexes on X to define $THH(X)$

We can use the full subcategory of U-acyclics to define $THH(X \text{ on } Y)$

Theorem (Localization for open subschemes)

There is a cofibration sequence of spectra

$$THH(X \text{ on } Y) \to THH(X) \to THH(U)$$

Theorem (Mayer-Vietoris)

There is a cofibration sequence of spectra

$$THH(U \cap V) \to THH(U) \vee THH(V) \to THH(U \cup V)$$

Conseqences

We can use a spectral model of the derived category of perfect complexes on X to define $THH(X)$

We can use the full subcategory of U-acyclics to define $THH(X \text{ on } Y)$

Theorem (Localization for open subschemes)

There is a cofibration sequence of spectra

$$THH(X \text{ on } Y) \to THH(X) \to THH(U)$$

Theorem (Mayer-Vietoris)

There is a cofibration sequence of spectra

$$THH(U \cap V) \to THH(U) \vee THH(V) \to THH(U \cup V)$$

Conseqences

We can use a spectral model of the derived category of perfect complexes on X to define $THH(X)$

We can use the full subcategory of U-acyclics to define $THH(X \text{ on } Y)$

Theorem (Localization for open subschemes)

There is a cofibration sequence of spectra

$$THH(X \text{ on } Y) \to THH(X) \to THH(U)$$

Theorem (Mayer-Vietoris)

There is a cofibration sequence of spectra

$$THH(U \cap V) \to THH(U) \vee THH(V) \to THH(U \cup V)$$

Concluding remarks

In the case of a quasi-projective scheme (or more generally a scheme with an ample family of line bundles), the bounded derived category is precisely the thick subcategory generated by the vector bundles.

The exact category of vector bundles, made into a spectral category, is the connective cover of the (full subcategory) spectral category we use above. Algebraic-geometric remarks aside, the difference between the last approach above and the first two approaches is using the full non-connective mapping spectra.

It turns out that for a connective ring spectrum R, forming the spectral category using the correct non-connective mapping spectra gives the same *THH* as the connective-cover spectral category. But this is another paper and another talk. . .

not in paper