Localization Sequences in $THH$ and $TC$

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Localization Sequences in $THH$ and $TC$

- Joint work with Andrew Blumberg
Overview

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Goal: Prove the analogue of the Thomason-Trobaugh $K$-theory Mayer-Vietoris and localization theorems in $THH$ and $TC$
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Strategy: Prove the analogue of Keller’s Hochschild homology and cyclic homology localization theorems for $THH$ and $TC$
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Recurring theme: connective vs. non-connective ring spectra
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Ocurring theme: connective vs. non-connective ring spectra
Cortiñas, Haesemeyer, Schlichting, and Weibel: $K$-theory and singularities. Proved:

- Weibel conjecture: $K_{-n}X = 0$ for $n > \dim(X)$
- Vorst conjecture: $X$ is regular if and only if $K_{\dim(X)+1}X \to K_{\dim(X)+1}(X \times \mathbb{A}^r)$ is an isomorphism for all $r > 0$.

Over fields of characteristic zero using Mayer Vietoris and localization in negative cyclic homology.

Geisser and Hesselholt proved Weibel conjecture in characteristic $p$ using corresponding $TC$ results (assuming strong resolution of singularities).
Motivation

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Hochschild Homology

Cyclic bar construction

$$N^c_y R = \underbrace{R \otimes \cdots \otimes R \otimes R}_{q \text{ factors}}$$

$$R \otimes \cdots \otimes R$$

$$R$$

$$HH_* R$$ is the homology of the resulting chain complex
Cyclic bar construction

\[ N^c_y HR = \underbrace{HR \wedge \cdots \wedge HR}^{q \text{ factors}} \wedge HR \]

\[ HR \wedge \cdots \wedge HR \wedge HR \]

THH(\(R\)) is the resulting spectrum
Morita Invariance

Both $HH$ and $THH$ have Morita invariance:

$$HH(R) \simeq HH(M_nR)$$
$$THH(R) \simeq THH(M_nR)$$

$\implies$ Dennis and cyclotomic trace maps
Dennis Trace / Cyclotomic Trace

Map $BGL_n R \to B^{cy} GL_n R$: 

$$B_q(GL_n R) = \underbrace{GL_n R \times \cdots \times GL_n R}_{q \text{ factors}}$$

$$B_q^{cy}(GL_n R) = \underbrace{GL_n R \times \cdots \times GL_n R \times GL_n R}_{q \text{ factors}}$$

by $(g_1 | \cdots | g_q) \mapsto (g_1 | \cdots | g_q)g_q^{-1} \cdots g_1^{-1}$.

Map $B^{cy} GL_n R \to N^{cy}(M_n R)$ or to $N^{cy}_q(HM_n R)$.

$$N_q^{cy}(M_n R) = \underbrace{M_n R \otimes \cdots \otimes M_n R \otimes M_n R}_{q \text{ factors}}$$

Fit together to a map $KR = (\coprod BGL_n R)^+ \to THH(R) \to HH(R)$. 
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Why $THH$?

- Relatively easy to compute
- Stabilization of $K$-theory [Dundas-McCarthy]

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But…

Not really that close to $K$-theory

$\Rightarrow$ $HN$ and $TC$
Negative Cyclic and $TC$

- Built from $HH$ and $THH$
- Still reasonably computable
- Goodwillie: Relative $HN$ is rationally equivalent to relative $K$-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative $TC$ is $p$-equivalent to relative $K$-theory (for surjective maps with nilpotent kernel).
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Homotopy cartesian square

$$\begin{array}{ccc}
K(R)_\mathbb{Q} & \rightarrow & K(R')_\mathbb{Q} \\
\downarrow & & \downarrow \\
HN(R \otimes \mathbb{Q}) & \rightarrow & HN(R' \otimes \mathbb{Q})
\end{array}$$

For $R \rightarrow R'$

Surjective map with nilpotent kernel.
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\end{align*}
$$

$$
\begin{align*}
K(R)^\wedge & \longrightarrow K(R')^\wedge \\
\downarrow & \downarrow \\
TC(R, p) & \longrightarrow TC(R', p)
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- Goodwillie: Relative $HN$ is rationally equivalent to relative $K$-theory (for surjective maps with nilpotent kernel).
- McCarthy: Relative $TC$ is $p$-equivalent to relative $K$-theory (for surjective maps with nilpotent kernel).
- Dundas: Generalized $p$-equivalence to maps of ring spectra.

Homotopy cartesian square

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K(R) & \rightarrow & K(R') \\
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TC(R, p) & \rightarrow & TC(R', p) \\
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\]
Quillen’s $K$-Theory of Schemes

$K(X) = K$-theory of exact category of vector bundles on $X$.

Algebraic-geometric Remarks:

- Definitively correct for quasi-projective varieties.
- “Obviously” not correct for varieties that do not have enough vector bundles.

- Formulation of perfect complex: A complex of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to bounded complex of vector bundles.
- Bounded derived category = full subcategory of derived category of $\mathcal{O}_X$-modules consisting of the perfect complexes.
- Bounded derived category is also the full subcategory of compact objects. [An alternative characterization of perfect complexes.]
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Thomason’s $K$-Theory of Schemes

Basic Outline

- Apply Waldhausen’s construction, which generalizes Quillen’s from exact categories to categories with cofibrations and weak equivalences.

- Work with Complicial biWaldhausen categories and functors: Subcategories of categories of complexes on abelian categories (with restrictions); functors induced by additive functors on the underlying abelian categories.

This gives a $K$-theory of derived categories (of sorts):

**Theorem.** If a complicial functor between complicial biWaldhausen categories induces an equivalence of derived categories, it induces an equivalence of $K$-theory.
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Thomason’s $K$-Theory of Schemes

Consequence: Any subcategory (with restrictions) of perfect complexes of $\mathcal{O}_X$-modules whose derived category is the bounded derived category produces the same $K$-theory.

\[ K(X) = K\text{-theory of the category of perfect complexes on } X. \]

Variant: For $Y$ a closed subset of $X$

\[ K(X \text{ on } Y) = K\text{-theory of the category of perfect complexes on } X \text{ that are supported on } Y \text{ (acyclic off } Y). \]
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**Thomason Trobaugh Localization Theorem**

**Localization Theorem.** Let $U \subset X$ be open, $Y = X - U$. There is a long exact sequence

\[ \cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(U) \to \cdots \]

\[ \cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U) \to \cdots \]

**Mayer-Vietoris Theorem.** Let $U, V \subset X$ with $X = U \cup V$. There is a long exact sequence

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Proof

Waldhausen: Fibration sequences for “weaker” weak equivalences.

A category with a weak equivalences \( w \), and another collection of weak equivalences \( v \) with \( v \subset w \).

\( \mathcal{C}^w \) = the subcategory of \( \mathcal{C} \) of objects \( w \)-equivalent to the trivial object.

**Theorem.** (Waldhausen localization sequence)
The following square is homotopy cartesian:

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\begin{array}{ccc}
K(\mathcal{C}^w, v) & \longrightarrow & K(\mathcal{C}^w, w) \\
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\( K(\mathcal{C}^w, w) \) is trivial.
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Take $\mathcal{C} = \text{Perfect complexes on } X$

$\nu = \text{Quasi-isomorphisms}$

$\omega = \text{Maps that are quasi-isomorphisms on } U$

$\nu \subset \omega$

$\mathcal{C}^\omega = \text{Perfect complexes supported on } Y.$

Thomason and Trobaugh prove that the derived category $\mathcal{D}(\mathcal{C}, \omega)$ is cofinal in the bounded derived category of $U$. 

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Neeman’s Abstract Reformulation

Let $S$ be a triangulated category generated by its compact objects $S^c$ and is closed under small coproducts and assume $S^c$ is small.

Let $\mathcal{R}$ be localizing subcategory gen. by some set of compact objects.

Let $\mathcal{T}$ be the triangulated quotient $S/\mathcal{R}$.

This means $\mathcal{T}$ is the localization of $S$ with respect to the maps whose cofibers are in $\mathcal{R}$.

**Theorem**

- The compact objects $S^c$ of $S$ map to compact objects of $\mathcal{T}$.
- The induced functor $S^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful and $\mathcal{T}^c$ is the thick subcategory generated by its image.

T-T localization set-up can be reformulated in terms of quotients of triangulated categories.
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Remark on Cofinality

Cofinality implies iso on $K_n$ for $n > 0$ but only an injection on $K_0$. This is why the localization sequence is generally not surjective on $K_0$:

$$
\cdots \to K_n(X \text{ on } Y) \to K_n(X) \to K_n(U) \to \cdots
$$

$$
\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U)
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The sequence actually continues with the Bass negative $K$-groups:

$$
\cdots \to K_0(X \text{ on } Y) \to K_0(X) \to K_0(U) \to K_{-1}(X \text{ on } Y) \to K_{-1}(X) \to \cdots
$$

These groups are defined inductively by

$$
K_{-n-1}X = \text{Coker } (K_{-n}(X \times \text{Spec } \mathbb{Z}[x]) \oplus K_{-n}(X \times \text{Spec } \mathbb{Z}[x^{-1}])
$$

$$
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Thomason and Trobaugh construct a non-connective $K$-theory spectrum $K^B X$ essentially by doing Bass’ algebraic construction on the spectrum level.

In terms of $K^B$, the localization theorem asserts a cofiber sequence of spectra

$$K^B(X \text{ on } Y) \to K^B(X) \to K^B(U)$$

and the Mayer-Vietoris theorem asserts a cofiber sequence of spectra

$$K^B(U \cap V) \to K^B(U) \vee K^B(V) \to K^B(U \cup V)$$
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Thomason and Trobaugh construct a non-connective $K$-theory spectrum $K^B X$ essentially by doing Bass’ algebraic construction on the spectrum level.

In terms of $K^B$, the localization theorem asserts a cofiber sequence of spectra

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First Try: Hochschild-Mitchell construction

For an additive category $\mathcal{C}$

$$\mathcal{N}^{cy}_q \mathcal{C} = \bigoplus_{x_0, \ldots, x_q \in \mathcal{C}} \mathcal{C}(x_q, x_{q-1}) \otimes \cdots \otimes \mathcal{C}(x_1, x_0) \otimes \mathcal{C}(x_0, x_q)$$

Constructs $HH(\mathcal{C})$.

Using (bar construction) Eilenberg-Mac Lane spectra, we get a spectral category $\mathcal{C}^S$.

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We could apply this to the category of vector bundles.
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Solution:
Mix Waldhausen’s $S\bullet$-construction in with the Hochschild-Mitchell construction.

Nice consequence:
Can reformulate cyclotomic trace as inclusion of objects in Hochschild-Mitchell construction.
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Mayer-Vietoris

The Dundas-McCarthy construction cannot satisfy Mayer-Vietoris.

Example. Look at the projective (elliptic) curve

\[ x_0 x_2^2 = x_1^3 - 3x_0^2 x_1 \]

This has an open cover by the affines

\[ U = \{ x_0 \neq 0 \} = \text{Spec } \mathbb{Z}[x, y]/(y^2 = x^3 - 3x) \]
\[ V = \{ x_2 \neq 0 \} = \text{Spec } \mathbb{Z}[u, v]/(u = v^3 - 3u^2 v) \]

for \( x = x_1/x_0, \ y = x_2/x_0, \ u = x_0/x_2, \ v = x_1/x_2 \)

Then \( U \cap V = \text{Spec } \mathbb{Z}[x, y, y^{-1}]/(y^2 = x^3 - 3x), \) but

\[ \mathbb{Z}[x, y]/(y^2 = x^3 - 3x) \oplus \mathbb{Z}[u, v]/(u = v^3 - 3u^2 v) \]
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is not surjective. (Here \( u \mapsto 1/y \) and \( v \mapsto x/y. \))
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Bass Construction

This is not related to the Bass construction.

No negative Bass $THH$ groups for rings:

$$\text{Coker} \left( THH_0(R[x]) \oplus THH_0(R[x^{-1}]) \rightarrow THH_0(R[x, x^{-1}]) \right)$$

is always surjective. It is

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Third Try: Geisser-Hesselholt

$THH$ of rings localizes: $\pi_* THH(R[S^{-1}]) = \pi_* THH(R) \otimes R[S^{-1}]$.

In other words, for a ring $\pi_* THH(R)$ is a quasi-coherent sheaf

Define $THH(X)$ as the Čech spectrum of an affine open cover, or as the hyper-cohomology spectrum.

Tautologically satisfies Mayer-Vietoris.

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**Construction for \( HH \): Keller**

Force localization to hold:

Setup: \( S \) a DG-category, \( \mathcal{R} \) a subcategory.

E.g., \( S \) a category of complexes, \( \mathcal{R} \) the acyclics.

Define: \( HH(S, \mathcal{R}) \) as the cofiber of Hochschild-Mitchell constructions

\[
HH(S, \mathcal{R}) = \text{Cofiber}(HH(\mathcal{R}) \rightarrow HH(S))
\]

(Definition actually due to Kassel.)

Keller then proves (roughly) that a map \((S, \mathcal{R}) \rightarrow (S', \mathcal{R}')\) that induces an equivalence on triangulated quotients induces an equivalence on \( HH \).
Our Approach: Concepts

Work with spectral categories and use the Hochschild-Mitchell complex (actually, the analogue due to Bokstedt).

A spectral category has an associated \( \text{homotopy category} \) defined by \( \pi_0 \) of the mapping spectra, or graded homotopy category defined by \( \pi_* \) of the mapping spectra.

A \textit{pretriangulated} spectral category is (roughly) a spectral category whose homotopy category is triangulated.

General theory shows that we can enhance a DG-category into a spectral category.
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DK-Invariance

Basic kind of equivalence of spectral categories: Dwyer-Kan equivalence.

A DK-equivalence of spectral categories is a spectral functor that is a weak equivalence on mapping spectra and an equivalence on the homotopy category.

Theorem

A DK-equivalence of spectral categories induces a weak equivalence of this THH.
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Morita Invariance / Cofinality

Up to DK-equivalence any (small) spectral category embeds in a pretriangulated spectral category.
We use this to simplify statements

**Theorem**

Let \( C \subseteq C' \) be full subcategories of the pretriangulated spectral category \( D \) with the objects of \( C' \) contained in the thick subcategory generated by the objects of \( C \) (in the triangulated category \( \pi_0 D \)). Then

\[
THH(C) \to THH(C')
\]

is a weak equivalence.
Localization Theorem

Let $A \subset B$ and $A' \subset B'$ be inclusions of full spectral categories and assume that they are all pretriangulated. Let $f : B \to B'$ be a spectral functor that restricts to $A \to A'$.

Theorem (Abstract localization theorem)

If the induced map of triangulated quotients is an equivalence then the map of cofibers

$$\text{Cofiber}(\text{THH}(A) \to \text{THH}(B)) \to \text{Cofiber}(\text{THH}(A') \to \text{THH}(B'))$$

is an equivalence.
**Consequences**

We can use a spectral model of the derived category of perfect complexes on $X$ to define $THH(X)$.

We can use the full subcategory of $U$-acyclics to define $THH(X_{\text{on}} Y)$.

**Theorem (Localization for open subschemes)**

There is a cofibration sequence of spectra

$$THH(X_{\text{on}} Y) \rightarrow THH(X) \rightarrow THH(U)$$

**Theorem (Mayer-Vietoris)**

There is a cofibration sequence of spectra

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Concluding remarks

In the case of a quasi-projective scheme (or more generally a scheme with an ample family of line bundles), the bounded derived category is precisely the thick subcategory generated by the vector bundles.

The exact category of vector bundles, made into a spectral category, is the connective cover of the (full subcategory) spectral category we use above. Algebraic-geometric remarks aside, the difference between the last approach above and the first two approaches is using the full non-connective mapping spectra.

It turns out that for a connective ring spectrum $R$, forming the spectral category using the correct non-connective mapping spectra gives the same $THH$ as the connective-cover spectral category. But this is another paper and another talk...